Star product realizations of κ-Minkowski space

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Abstract

We define a family of star products and involutions associated with κ-Minkowski space. Applying corresponding quantization maps we show that these star products restricted to a certain space of Schwartz functions have isomorphic Banach algebra completions. For two particular star products it is demonstrated that they can be extended to a class of polynomially bounded smooth functions allowing a realization of the full Hopf algebra structure on κ-Minkowski space. Furthermore, we give an explicit realization of the action of the κ-Poincaré algebra as an involutive Hopf algebra on this representation of κ-Minkowski space and initiate a study of its properties.

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1 Introduction

The κ-deformation of Minkowski space was originally proposed in [10] as a Hopf algebra whose underlying algebra is the enveloping algebra of the Lie algebra with generators $x_0, \ldots, x_{d-1}$ fulfilling

$$[x_0,\, x_i] = \tfrac{i}{\kappa}\, x_i, \quad [x_i,\, x_j] = 0, \quad i,j = 1,\ldots, d-1, \tag{1}$$

where $\kappa \neq 0$ can be viewed as a deformation parameter, since formally, in the limit $\kappa \to \infty$ one obtains the commutative coordinate algebra of Minkowski space. Of course, to single out this limit as the Minkowski space requires some additional structure involving the action of the

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Poincaré group, or rather a deformed version thereof, for finite κ [9]. This was, originally, how the algebra was conceived and we shall return to this issue in Section 4. For the moment we concentrate on (1).

The first object of this paper is to discuss a class of star-products on $\mathbb{R}^d$ and associated quantization maps based on the harmonic analysis on the Lie group associated with (1). The motivation originates from a similar approach to the standard Weyl quantization map based on its relation to the Heisenberg algebra of quantum mechanics. For the purpose of later reference let us briefly recall the main steps in this construction. The Heisenberg algebra associated to a particle moving on the real line is the three-dimensional Lie algebra defined by the relation

$$[P, Q] = iC,$$

where C is a central element. The real form of this algebra with basis iP, iQ, iC has a faithful representation σ in terms of strictly upper triangular matrices:

$$\sigma(i(aP + bQ + cC)) = \begin{pmatrix} 0 & a & c \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix}. \tag{2}$$

The connected and simply connected Lie group of the algebra is by definition the *Heisenberg group*, which we denote by $\mathcal{H}eis$. It is the group of upper triangular matrices of the form

$$T(a, b, c) = \begin{pmatrix} 1 & a & c + \frac{1}{2}ab \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}, \tag{3}$$

which is obtained by exponentiation of (2). The group operations, expressed in this parametrization, are seen to be

$$T(a, b, c)T(a', b', c') = T(a + a', b + b', c + c' + \frac{1}{2}(ab' - a'b)), \tag{4}$$

$$T(a, b, c)^{-1} = T(-a, -b, -c). \tag{5}$$

It follows that $\mathcal{H}eis$ is a unimodular group with Haar measure equal to $dadbdc$. Thus the group algebra of $\mathcal{H}eis$ can be identified with $L^1(\mathbb{R}^3)$ via the parametrization (3). Let $\circ$ denote the convolution product on the group algebra.

According to the Stone-von Neumann theorem [11] the non-trivial irreducible unitary representations of $\mathcal{H}eis$ are labelled by the value $\hbar \neq 0$ of the central element C. Fixing $\hbar$, the representation π can be expressed in the form

$$\pi(T(a, b, c)) = e^{\frac{i}{2}\hbar c}U(a, b), \tag{6}$$

$$(U(a, b)\psi)(x) = e^{\frac{i}{2}\hbar ab}e^{ibx}\psi(x - \hbar a), \tag{7}$$

for $\psi \in L^2(\mathbb{R})$. The corresponding representation of the group algebra, also denoted by π, is then given by

$$\pi(F) = \int_{\mathbb{R}^3} dadbdc F(a, b, c)\pi(T(a, b, c)) = \int_{\mathbb{R}^2} F^\sharp(a, b)\, U(a, b), \tag{8}$$

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where

$$F^\sharp(a, b) = \int dc F(a, b, c) e^{-i\hbar c}.$$

Clearly, $F \to F^\sharp$ maps $\mathrm{L}^1(\mathbb{R}^3)$ onto $\mathrm{L}^1(\mathbb{R}^2)$ and a simple calculation yields

$$(F \circ G)^\sharp(a, b) = \int_{\mathbb{R}^2} da' db' F^\sharp(a, b) G^\sharp(a - a', b - b') e^{\frac{i}{2}(ab' - a'b)}, \tag{9}$$

where the last expression is a "twisted" convolution product on $\mathbb{R}^2$ that we shall denote by $F^\sharp \hat\circ G^\sharp$, and where we have set $\hbar = 1$ for the sake of simplicity.

According to (8) we may write $\pi(f)$ instead of $\pi(F)$ when $f = F^\sharp$. With this notation the Weyl quantization map W is defined as

$$W(f) = \pi(\mathcal{F}f), \tag{10}$$

for $f \in \mathrm{L}^1(\mathbb{R}^2) \cap \mathcal{F}^{-1}(\mathrm{L}^1(\mathbb{R}^2))$, where $\mathcal{F}$ denotes the Fourier transform on $\mathbb{R}^2$,

$$(\mathcal{F}f)(a, b) = \frac{1}{2\pi} \int d\alpha d\beta \, f(\alpha, \beta) e^{-i(a\alpha + b\beta)}. \tag{11}$$

Using

$$\pi(F \circ G) = \pi(F)\pi(G) \qquad \text{for } F, G \in \mathrm{L}^1(\mathbb{R}^3),$$

we obtain from (8) and (9) that

$$W(f *_0 g) = W(f)W(g),$$

where f and g are functions on $\mathbb{R}^2$ and their *Weyl-product* is given by

$$f *_0 g(\alpha, \beta) = \mathcal{F}^{-1}(\mathcal{F}f) \hat\circ (\mathcal{F}g)), \tag{12}$$

which clearly is well defined when f and g are Schwartz functions.

From this definition the familiar expressions (see e.g. [7]) for the Weyl product can easily be derived. Likewise, the Weyl operators $W(f)$ can be seen to be integral operators for appropriate functions f. In particular, it can be shown that $W(f)$ is of Hilbert-Schmidt type if and only if f is square integrable, and in this case

$$\|W(f)\|_2^2 = 2\pi \int d\alpha d\beta |f(\alpha, \beta)|^2, \tag{13}$$

where $\|\cdot\|$ denotes the Hilbert-Schmidt norm. It follows that the Weyl product can be extended to square integrable functions and W can be extended to an isomorphism between the resulting algebra and the Hilbert-Schmidt operators on $\mathrm{L}^2(\mathbb{R})$.

It is worth emphasizing that the construction outlined here depends on the chosen parametrization (3). An alternative parametrization preserving the invariant measure is, e.g.,

$$(a, b, c) \to T(a, b, c + \xi ab),$$

where ξ is a real constant. In this case, one obtains a quantization map W_ξ and a star product $*_\xi$ that are related to $*_0$ by

$$W_\xi(f) = W(\Psi_\xi f), \qquad \Psi_\xi(f *_\xi g)) = (\Psi_\xi f) *_0 (\Psi_\xi g),$$

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where Ψ_ξ is defined by

$$(\Psi_\xi f)(\alpha, \beta) = e^{i\xi\alpha\beta} f(\alpha, \beta)\,.$$

It follows that Ψ_ξ is an isomorphism of star-algebras of Schwartz functions and, moreover, since both $\mathcal{F}$ and multiplication by a phase factor preserve the norm in $\mathrm{L}^2(\mathbb{R}^2)$ we have that (13) is also fulfilled with W replaced by W_ξ. In particular, one can verify that $W_{-\frac{1}{2}}$ is the so-called Kohn-Nirenberg quantization map, in which case $W_{-\frac{1}{2}}(f)$ is the pseudo-differential operator with symbol f. The Weyl map is singled our among the maps W_ξ by the property

$$W(f)^* = W(\bar{f})\,,$$

where $\bar{f}$ is the complex conjugate of f.

The purpose of this paper to is to develop an approach similar to the preceding to quantization maps associated with κ-Minkowski space for $d = 2$, which we denote by M_κ. In Section 2 we introduce the κ-Minkowski group G, analogous to $\mathcal{H}eis$, and via harmonic analysis on G we define a family of products, called *star products*, and involutions for a class $\mathcal{B}$ of Schwartz functions on $\mathbb{R}^2$. Explicit expressions for the star products and operator kernels are obtained which are used to show that those involutive algebras have natural isomorphic Banach algebra completions. In Section 3 two particular star products associated to the left and right invariant Haar measures on G are discussed. It is shown that they have natural extensions to a certain sub-algebra $\mathcal{C}$ of the multiplier algebra of $\mathcal{B}$ consisting of smooth functions of polynomial growth. Moreover, it is shown that the resulting algebra has a Hopf star algebra structure furnishing a star product representation of κ-Minkowski space. In Section 4 we show how to represent the action of the κ-Poincaré algebra $\mathcal{P}_\kappa$ on κ-Minkowski space in this particular realization as well as on the subalgebra $\mathcal{B}$. On the latter we show that the Lebesgue integral is a twisted trace, invariant under the action of $\mathcal{P}_\kappa$. Finally, Section 5 contains some concluding remarks and a few technical details are collected in an appendix.

2 Quantizations and star products

2.1 The right-invariant case

In the following we restrict attention to $d = 2$ in which case the Lie algebra defined by (1) is the unique noncommutative Lie algebra of dimension 2 and κ-Minkowski space M_κ is its universal enveloping algebra. We set $x = x_1$ and $t = \kappa x_0$ and consider the real form of the Lie algebra with generators it, ix fulfilling

$$[t, x] = ix. \tag{14}$$

It has a faithful 2-dimensional representation ρ given by

$$\rho(it) = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \rho(ix) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \tag{15}$$

and the corresponding connected and simply connected Lie group is the group G of 2×2-matrices of the form

$$S(a, b) = \begin{pmatrix} e^{-a} & b \\ 0 & 1 \end{pmatrix}, \quad a, b \in \mathbb{R}, \tag{16}$$

obtained by exponentiating ρ:

$$e^{i\rho(at+b'x)} = \begin{pmatrix} e^{-a} & \frac{1-e^{-a}}{a}b' \\ 0 & 1 \end{pmatrix}. \tag{17}$$

The group operations written in the (a, b) coordinates become

$$S(a_1, b_1)S(a_2, b_2) = S(a_1 + a_2, b_1 + e^{-a_1}b_2), \qquad S(a, b)^{-1} = S(-a, -e^a b). \tag{18}$$

An immediate consequence is

Lemma 2.1. *The Lebesgue measure $da\,db$ is right invariant whereas the measure $e^a da\,db$ is left-invariant on G. In particular, G is not unimodular.*

Let $\mathcal{A}$ denote the convolution algebra of G with respect to the right invariant measure. Identifying functions on G with functions on $\mathbb{R}^2$ by the parametrization (16) then $\mathcal{A}$ is the involutive Banach algebra consisting of integrable functions on $\mathbb{R}^2$ with product $\hat{*}$ and involution † given by

$$(f \hat{*} g)(a, b) = \int da' db' f(a - a', b - e^{a'-a}b')g(a', b'), \tag{19}$$

$$f^\dagger(a, b) = e^a \bar{f}(-a, -e^a b), \tag{20}$$

where $f, g \in \mathcal{A}$ and $\bar{f}$ is the complex conjugate of f. If π is a unitary representation of G (always assumed to be strongly continuous in the following) it is well known (see e.g. [12]) that π gives rise to a representation, also denoted by π, of $\mathcal{A}$ by setting

$$\pi(f) = \int da db\, f(a, b)\pi(S(a, b)). \tag{21}$$

Thus, we have

$$\pi(f \hat{*} g) = \pi(f)\pi(g) \quad \text{and} \quad \pi(f^\dagger) = \pi(f)^*. \tag{22}$$

Following the same procedure as described for the Weyl quantization above we define the Weyl map W_π associated with the representation π by

$$W_\pi(f) = \pi(\mathcal{F}f) \ \text{ for } f \in \mathrm{L}^1(\mathbb{R}^2) \cap \mathcal{F}^{-1}(\mathrm{L}^1(\mathbb{R}^2)),$$

where $\mathcal{F}$ denotes the Fourier transform (11) on $\mathbb{R}^2$. It then follows from (22) that

$$W_\pi(f * g) = W_\pi(f)W_\pi(g) \quad \text{and} \quad W_\pi(f^*) = W_\pi(f)^*$$

where the $*$-product and the $*$-involution are defined by

$$f * g = \mathcal{F}^{-1}\left((\mathcal{F}f)\hat{*}(\mathcal{F}g)\right). \tag{23}$$

and

$$f^* = \mathcal{F}^{-1}(\mathcal{F}(f)^\dagger)\,, \tag{24}$$

respectively. As in the case of the standard Moyal product, one needs to exercise care about the domain of definition for the right-hand sides of (23) and (24). I this section we restrict our attention to the subset $\mathcal{B}$ of Schwartz functions introduced in the following definition, while an extension to a class of polynomially bounded functions will be discussed in subsequent sections.

Definition 2.2. *Let $\mathcal{S}_c$ denote the space of Schwartz functions on $\mathbb{R}^2$ with compact support in the first variable, i.e., $\mathrm{supp}(f) \subseteq K \times \mathbb{R}$, where $K \subseteq \mathbb{R}$ is compact. Then we define $\mathcal{B} = \mathcal{F}(\mathcal{S}_c) = \mathcal{F}^{-1}(\mathcal{S}_c)$.*

Proposition 2.3. *If $f, g \in \mathcal{B}$ then f^* and $f * g$ also belong to $\mathcal{B}$ and are given by*

$$f * g(\alpha, \beta) = \frac{1}{2\pi} \int dv \int d\alpha'\, f(\alpha + \alpha', \beta) g(\alpha, e^{-v}\beta) e^{-i\alpha'v}, \tag{25}$$

and

$$f^*(\alpha, \beta) = \frac{1}{2\pi} \int dv \int d\alpha'\, \bar{f}(\alpha + \alpha', e^{-v}\beta) e^{-i\alpha'v}. \tag{26}$$

respectively.

Proof. Invariance of $\mathcal{F}^{-1}(\mathcal{S}_c)$ under the $*$-product and the *-involution follows from the fact that $\mathcal{S}_c$ is an involutive subalgebra of the convolution algebra $\mathcal{A}$ as is easily seen from (19) and (20).

In order to establish (25), note that its right-hand side equals

$$\frac{1}{\sqrt{2\pi}} \int dv \tilde{f}(v, \beta) g(\alpha, e^{-v}\beta) e^{i\alpha v}\,, \tag{27}$$

where $\tilde{f}$ denotes the Fourier transform of f w.r.t. the first variable

$$\tilde{f}(a, \beta) = \frac{1}{\sqrt{2\pi}} \int d\alpha\, f(\alpha, \beta) e^{-ia\alpha}.$$

Note that the integrand in (27) is a Schwartz function of v, α, β with compact support in v. Thus it suffices to show that the Fourier transform of (27) w.r.t. α, β equals $\mathcal{F}f \hat{*} \mathcal{F}g$. This follows from a straightforward calculation using the Plancherel theorem on the β-integral.

Concerning (26) we note similarly that the right-hand side equals

$$\frac{1}{\sqrt{2\pi}} \int dv \bar{\tilde{f}}(-v, e^{-v}\beta) e^{i\alpha v}\,. \tag{28}$$

Here, we note that the integrand is a Schwartz function of v, β, such that Fourier transforming (28) w.r.t. β gives

$$\frac{1}{\sqrt{2\pi}} \int dv \overline{\mathcal{F}f}(-v, -e^v b) e^v e^{i\alpha v} = \frac{1}{\sqrt{2\pi}} \int dv (\mathcal{F}f)^\dagger(v, b) e^{i\alpha v}\,. \tag{29}$$

Hence, by Fourier inversion and (24) we conclude that the Fourier transform of the right-hand side of (26) equals $\mathcal{F}(f^*)$. This proves (26). □

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Note that associativity of the above defined star product on $\mathcal{B}$ is an immediate consequence of associativity of the convolution product on $\mathcal{A}$. Likewise, $f \to f^*$ is an involution on $\mathcal{B}$, since $f \to f^\dagger$ is an involution on $\mathcal{A}$. Thus we have

Corollary 2.4. $\mathcal{B}$ *equipped with the* $*$-*product and* *-*involution defined by* (25) *and* (26) *is an involutive algebra.*

It should be noted that the star product and the involution as defined by (23) and (24) are independent of the choice of representation π of G, while the quantization map W_π, that we proceed to discuss next, is indeed representation dependent. G being isomorphic to the identity component of the group of affine transformations on $\mathbb{R}$, its representation theory is well known [8]. In particular, there is a close relationship to the representation theory of the Heisenberg group [1]. The basic result we shall use is the following, the proof of which is included for the sake of completeness (see also [2]).

Proposition 2.5. G *has exactly two non-trivial unitary representations* $\pi_\pm$. *Their action on the generators* t, x *is given by*

$$\pi_+(t) \;=\; -i\frac{d}{ds}\,, \qquad \pi_+(x) = e^{-s}\,, \tag{30}$$

$$\pi_-(t) \;=\; -i\frac{d}{ds}\,, \qquad \pi_-(x) = -e^{-s}\,, \tag{31}$$

as self-adjoint operators on $\mathrm{L}^2(\mathbb{R})$.
All other irreducible unitary representations are one-dimensional of the form $\pi_c(x) = 0$ *and* $\pi_c(t) = c$ *for some* $c \in \mathbb{R}$.

Proof. Let π be a unitary representation of G on a Hilbert space H and let $v \in Dom(\pi(x))$. Differentiating the relation

$$e^{i a \pi(t)} e^{i b \pi(x)} e^{-i a \pi(t)} v = e^{i b e^{-a} \pi(x)} v\,, \tag{32}$$

which follows from (18), w.r.t. b we get that $e^{-i a \pi(t)} v \in Dom(\pi(x))$ and

$$e^{i a \pi(t)} \pi(x) e^{-i a \pi(t)} v = e^{-a} \pi(x) v\,,$$

so the two self-adjoint operators $e^{i a \pi(t)} \pi(x) e^{-i a \pi(t)}$ and $e^{-a} \pi(x)$ coincide. But since $e^{-a} > 0$ the spectral subspaces H_+, H_- and H_0 corresponding to the positive and negative real line and $\{0\}$, respectively, are identical for $\pi(x)$ and $e^{i a \pi(t)} \pi(x) e^{-i a \pi(t)}$. It follows that those spaces are invariant under $e^{i a \pi(t)}$ and $e^{i b \pi(x)}$. By irreducibility one of them equals H and the other two vanish.
Assume $H = H_+$ and define the self-adjoint operator Q by

$$Q = -\ln(\pi(x))\,.$$

Then $x = e^{-Q}$ and by (32) we have

$$\exp\left(i b e^{i a \pi(t)} e^{-Q} e^{-i a \pi(t)}\right) = \exp\left(i b e^{-a} e^{-Q}\right)\,,$$

and hence
$$e^{ia\pi(t)}e^{-Q}e^{-ia\pi(t)} = \exp\left(e^{ia\pi(t)}Qe^{-ia\pi(t)}\right) = e^{-Q-a}\,.$$

Taking logarithms gives
$$e^{ia\pi(t)}Qe^{-ia\pi(t)} = Q + a$$

and consequently
$$e^{ia\pi(t)}e^{ibQ}e^{-ia\pi(t)} = e^{iab}e^{ibQ}\,.$$

which is recognized as the Weyl form of the canonical commutation relations. Applying the Stone-von Neumann theorem [11] we conclude that $\pi = \pi_+$. Similarly one shows that $\pi = \pi_-$ if $H = H_-$, and the case $H = H_0$ yields the one-dimensional representations as asserted. $\square$

We will use the notation $W_\pm$ for $W_{\pi_\pm}$. From the explicit form (6) of the action of the Heisenberg group in an irreducible representation one obtains the action of G in the representations $\pi_\pm$. Using $S(a,b) = S(0,b)S(a,0)$ the result is

$$\pi_\pm(S(a,b))\psi(s) = e^{\pm ibe^{-s}}\psi(s+a)\,, \quad \psi \in \mathrm{L}^2(\mathbb{R})\,.$$

It is now straightforward to determine the action of $W_\pm(f)$ for arbitrary $f \in \mathrm{L}^1(\mathbb{R}^2) \cap \mathcal{F}^{-1}(\mathrm{L}^1(\mathbb{R}^2))$. If $\langle \varphi, \psi \rangle$ denotes the inner product of $\varphi, \psi \in \mathrm{L}^2(\mathbb{R}^2)$ we get

$$
\begin{aligned}
\langle \varphi, W_\pm(f)\psi \rangle &= \int dadbds\, \mathcal{F}f(a,b)\,\overline{\varphi}(s)\,e^{\pm ibe^{-s}}\psi(s+a) \\
&= \int dsdudb\, \overline{\varphi}(s)\mathcal{F}f(u-s,b)\,e^{\pm ibe^{-s}}\psi(u) \\
&= \sqrt{2\pi}\int dsdu\, \overline{\varphi}(s)\tilde{f}(u-s,\pm e^{-s})\psi(u)\,.
\end{aligned}
$$

Hence we have shown

Proposition 2.6. *For $f \in \mathrm{L}^1(\mathbb{R}^2) \cap \mathcal{F}^{-1}(\mathrm{L}^1(\mathbb{R}^2))$ the operators $W_\pm(f)$ are integral operators on $\mathrm{L}^2(\mathbb{R})$ with kernels given by*

$$K_f^\pm(s,u) = \sqrt{2\pi}\tilde{f}(u-s,\pm e^{-s}) = \int dv f(v,\pm e^{-s})\,e^{-iv(u-s)}\,.$$

As a consequence we can establish the following basic identities.

Proposition 2.7. *a) $W_\pm(f)$ is of Hilbert-Schmidt type if and only if the restriction of f to $\mathbb{R} \times \mathbb{R}_\pm$ is square integrable w.r.t. the measure*

$$d\mu = |\beta|^{-1}d\alpha d\beta\,,$$

and we have

$$\|W_\pm(f)\|_2^2 = 2\pi \int_\mathbb{R} da \int_{\mathbb{R}_\pm} db |f(\alpha,\beta)|^2 \frac{d\alpha d\beta}{|\beta|} = 2\pi \int_{\mathbb{R}^2} dsdv |f(v,\pm e^{-s})|^2\,, \qquad (33)$$

where $\|\cdot\|_2$ denotes the Hilbert-Schmidt norm.

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b) If $W_\pm(f)$ *is trace class then*

$$tr W_\pm(f) = \int_{\mathbb{R}^2} dsdv f(v, \pm e^{-s}). \tag{34}$$

Proof. a) The operator $W_\pm(f)$ is Hilbert-Schmidt if and only if its kernel is square integrable. From Proposition 2.6 we get

$$\int dsdu |K_f^\pm(s,u)|^2 = 2\pi \int dsdu |\tilde{f}(u-s, \pm e^{-s})|^2 = 2\pi \int dsdu |\tilde{f}(u, \pm e^{-s})|^2.$$

Applying the Plancherel theorem on the u-integral then proves the first assertion as well as (33). b) If $W_\pm(f)$ is trace class, then

$$tr W_\pm(f) = \int_{\mathbb{R}^2} ds K_f^\pm(s,s),$$

and (34) follows from Proposition 2.6. $\qquad\square$

We note that although $\mathcal{B}$ is not contained in $L^2(\mathbb{R}^2, d\mu)$ we have that $\mathcal{B} \cap L^2(\mathbb{R}^2, d\mu)$ is dense in $L^2(\mathbb{R}^2, d\mu)$. Indeed, let $\mathcal{B}'$ denote the subspace of $\mathcal{B}$ consisting of Fourier transforms of derivatives w.r.t. the second variable of functions in $\mathcal{S}_c$. A function $f(\alpha, \beta)$ in $\mathcal{B}'$ is then of the form $\beta g(\alpha, \beta)$ where g is a Schwartz function, hence $f \in L^2(\mathbb{R}^2, d\mu)$. Moreover, if f is orthogonal to $\mathcal{B}'$ in $L^2(\mathbb{R}^2, d\mu)$ then its Fourier transform, considered as a tempered distribution, vanishes as a distribution, hence also as a tempered distribution. Thus $f = 0$ and we conclude that $\mathcal{B}'$ is dense in $L^2(\mathbb{R}^2, d\mu)$.

It follows from this remark and (33) that the mappings $W_\pm$ have unique extensions from $\mathcal{B}'$ to $L^2(\mathbb{R}^2, d\mu)$ such that (33) still holds. In particular, the map

$$W : f \to W_+(f) \oplus W_-(f)$$

is injective from $L^2(\mathbb{R}^2, d\mu)$ into $\mathcal{H} \oplus \mathcal{H}$, where $\mathcal{H}$ denotes the space of Hilbert-Schmidt operators on $L^2(\mathbb{R})$.

On the other hand, it is clear from the proof of Proposition 2.7 that any pair of kernels $K^\pm$ in $L^2(\mathbb{R}^2)$ originate from an $f \in L^2(\mathbb{R}^2, d\mu)$, i.e. W is unitary up to a factor $\sqrt{2\pi}$. This proves the following extension result.

Theorem 2.8. *Let $\mathcal{B}'$ and W be as defined above and set $\bar{\mathcal{B}} = L^2(\mathbb{R}^2, d\mu)$. Then the $*$-product* (25) *and involution* (26) *have unique extensions from $\mathcal{B}'$ to $\bar{\mathcal{B}}$, such that $\bar{\mathcal{B}}$ becomes a Banach algebra and W an isomorphism,*

$$W(f * g) = W(f)W(g) \qquad W(f^*) = W(f)^*.$$

If we complete the algebra $\bar{\mathcal{B}}$ in the operator norm, the resulting C^* algebra will be that of compact operators.

Corollary 2.9. *The integral w.r.t. $d\mu$ over $\mathbb{R} \times \mathbb{R}_\pm$ is a positive trace on $\bar{\mathcal{B}}$ in the following sense: for any $f, g \in \bar{\mathcal{B}}$,*

$$\int duds\,(f * f^*)(u, \pm e^{-s}) \geq 0 \quad and \quad \int duds(f * g)(u, \pm e^{-s}) = \int duds(g * f)(u, \pm e^{-s}).$$

Proof. If $f \in \bar{\mathcal{B}}$ then $W(f)$ is Hilbert-Schmidt and the first inequality follows from (33). If $f, g \in \bar{\mathcal{B}}$ then $W(f)W(g)$ is trace class and the second identity follows from Theorem 2.8 and (34). $\qquad\square$

For later use we note the following identities.

Proposition 2.10.   *a)   If $f, g \in \bar{\mathcal{B}}$ then*

$$\int d\alpha d\beta |\beta|^{-1}(f * g^*)(\alpha, \beta) = \int d\alpha d\beta |\beta|^{-1} f(\alpha, \beta)\, \bar{g}(\alpha, \beta)\,.$$

b)   If $f, g \in \mathcal{B}$ then

$$\int d\alpha d\beta\, (f * g^*)(\alpha, \beta) = \int d\alpha d\beta\, f(\alpha, \beta)\, \bar{g}(\alpha, \beta)\,, \tag{35}$$

$$\int d\alpha d\beta f^*(\alpha, \beta) = \int d\alpha d\beta \bar{f}(\alpha, \beta)\,. \tag{36}$$

Proof. a) Follows immediately from Proposition 2.7 and Theorem 2.8.
b) Using (23) and (24) as well as (19) and (20) we have

$$
\begin{aligned}
\int d\alpha d\beta\, (f * g^*)(\alpha, \beta) &= \mathcal{F}(f * g^*)(0) = (\mathcal{F}(f)\hat{*}\mathcal{F}(g^*))(0) \\
&= (\mathcal{F}(f)\hat{*}\mathcal{F}(g)^\dagger)(0) = \int dadb\, \mathcal{F}f(-a, -e^a b)\overline{\mathcal{F}g}(-a, -e^a b)e^a \\
&= \int dadb\, \mathcal{F}f(a, b)\overline{\mathcal{F}g}(a, b) = \int d\alpha d\beta\, f(\alpha, \beta)\, \bar{g}(\alpha, \beta)\,.
\end{aligned}
$$

Similarly, we have

$$\int d\alpha d\beta\, f^*(\alpha, \beta) = \mathcal{F}(f^*)(0) = (\mathcal{F}(f)^\dagger)(0) = \overline{\mathcal{F}(f)}(0) = \int d\alpha d\beta\, \bar{f}(\alpha, \beta)\,.$$

$\qquad\square$

In particular, it follows that

$$\int d\alpha d\beta\, (f * f^*)(\alpha, \beta) \geq 0\,, \quad f \in \mathcal{B}\,,$$

but in general $\int d\alpha d\beta\, f * g(\alpha, \beta) \neq \int d\alpha d\beta\, g * f(\alpha, \beta)$, i.e. $\int d\alpha d\beta$ is not a trace on $\mathcal{B}$. However, we shall see in Proposition 4.7 that $\int d\alpha d\beta$ satisfies a twisted trace property.

2.2   The left-invariant case and other star-products

The above procedure can be also applied to the convolution algebra of the left invariant measure on G instead of the right invariant one. It is then convenient to use the parametrization

$$R(a, c) = S(a, e^{-a}c)\,, \quad a, c \in \mathbb{R}\,, \tag{37}$$

in which the left invariant measure is $dadc$ by Lemma 2.1. Given a unitary representation π of G, the corresponding quantization map $\tilde{W}_\pi$ is defined by

$$\tilde{W}_\pi(f) = \int dadc\, \mathcal{F}f(a,c)\pi(R(a,c)) = \int dadc\, \mathcal{F}f(a,c)\pi(S(a,e^{-a}c)),$$

for $f \in \mathrm{L}^1(\mathbb{R}^2) \cap \mathcal{F}^{-1}(\mathrm{L}^1(\mathbb{R}^2))$. More generally, let us consider the map W_π^φ given by

$$W_\pi^\varphi(f) = \int dadc\, \mathcal{F}f(a,c)\pi(S(a,\varphi(a)c)), \tag{38}$$

where φ is a smooth, positive function on $\mathbb{R}$. Defining

$$(Uf)(a,b) = f(a,\eta(a)b)\,\eta(a),$$

for any function f of two variables, where

$$\eta(a) = \varphi(a)^{-1},$$

a change of variables in (38) gives

$$W_\pi^\varphi(f) = \pi(U\mathcal{F}f). \tag{39}$$

The corresponding star-product $*_\phi$ and involution $*^\varphi$ are given by

$$f *_\varphi g(\alpha,\beta) = \frac{1}{2\pi}\mathcal{F}^{-1}U^{-1}((U\mathcal{F}f)\,\hat{*}\,(U\mathcal{F}g)) \tag{40}$$

and

$$f^{*\varphi}(\alpha,\beta) = \mathcal{F}^{-1}U^{-1}((U\mathcal{F}f)^\dagger), \tag{41}$$

which are easily seen to be well defined for $f,g \in \mathcal{B}$. More explicitly, the following result holds.

Proposition 2.11. *If $f,g \in \mathcal{B}$ and φ is positive and smooth then*

$$f *_\varphi g(\alpha,\beta) = \frac{1}{2\pi}\int dadb\, \tilde{f}(b,\omega(a,b)e^{a-b}\beta)\tilde{g}(a-b,\omega(a,a-b)\beta)e^{i\alpha a}, \tag{42}$$

and

$$f^{*\varphi}(\alpha,\beta) = \frac{1}{2\pi}\int dv \int d\alpha'\, \bar{f}(\alpha+\alpha',\omega(a,-a)e^a\beta)e^{-i\alpha'v}, \tag{43}$$

where

$$\omega(a,b) = \eta(a)\varphi(b)e^{b-a}.$$

In particular, the star product $\star$ for the left-invariant measure, obtained for $\varphi(a) = e^{-a}$, becomes

$$f \star g(\alpha,\beta) = \frac{1}{2\pi}\int dv \int d\alpha'\, f(\alpha,e^v\beta)g(\alpha+\alpha',\beta)e^{-i\alpha'v}, \tag{44}$$

and the involution $\star$ for the left-invariant product is

$$f^\star(\alpha,\beta) = \frac{1}{2\pi}\int dv \int d\alpha'\, \bar{f}(\alpha+\alpha',e^v\beta)e^{-i\alpha'v}. \tag{45}$$

Proof. The first two identities follow by straightforward computation using (40) and (41) and Fourier inversion. The last two identities follow from the first two after a change of variables combined with Fourier inversion. Details are left to the reader. □

Definition 2.12. *By $\mathcal{B}_\varphi$ we shall denote the involutive algebra obtained by equipping $\mathcal{B}$ with the product $*_\varphi$ and involution $*^\varphi$.*

Remark 2.13. *In [6] a star product is obtained by a somewhat different approach involving a reducible representation of G acting on functions of two variables. Although the explicit form of that star product is not given in [6], it can be verified that it indeed coincides with (25).*
The star product considered in [2] (and in [3–5]) corresponds to the case $\varphi(a) = \frac{1-e^{-a}}{a}$ above and has the property that the involution equals complex conjugation. However, this property does not determine the star product uniquely among the products $*_\varphi$, as it holds more generally if φ satisfies the relation

$$\varphi(-a) = e^a \varphi(a), \quad a \in \mathbb{R}.$$

The form of the Weyl operators $W_\pm^\varphi(f)$ for $\pi = \pi_\pm$ is obtained from (39) and Proposition 2.6 by an easy computation that we omit. The result is the following.

Proposition 2.14. *Assume φ is positive and smooth. For $f \in \mathrm{L}^1(\mathbb{R}^2) \cap \mathcal{F}^{-1}(\mathrm{L}^1(\mathbb{R}^2))$ the operators $W_\pm^\varphi(f)$ are integral operators on $\mathrm{L}^2(\mathbb{R})$ with kernels given by*

$$K_f^\pm(s,u) = \sqrt{2\pi}\tilde{f}(u-s, \pm\varphi(u-s)e^{-s}) = \int dv f(v, \pm\varphi(u-s)e^{-s})e^{-iv(u-s)}.$$

It can now be seen that the norm and trace formulas (33) and (34) hold independently of the choice of φ.

Proposition 2.15.   *a) $W_\pm^\varphi(f)$ is Hilbert-Schmidt if and only if the restriction of f to $\mathbb{R} \times \mathbb{R}_\pm$ is square integrable w.r.t. the measure $d\mu$ and we have*

$$\|W_\pm^\varphi(f)\|_2^2 = 2\pi \int_{\mathbb{R}^2} dsdv |f(v, \pm e^{-s})|^2. \tag{46}$$

  b) If $W_\pm^\varphi(f)$ is trace class then

$$tr W_\pm^\varphi(f) = \int_{\mathbb{R}^2} dsdv f(v, \pm e^{-s}). \tag{47}$$

Proof. a) Using Proposition 2.14 we get

$$\begin{aligned}
\|W_\pm^\varphi(f)\|_2^2 &= 2\pi \int duds |\tilde{f}(u-s, \pm\varphi(u-s)e^{-s})|^2 \\
&= 2\pi \int dvds f(v, \pm\varphi(v)e^{-s})|^2 = 2\pi \int dvdr |f(v, \pm e^{-r})|^2
\end{aligned}$$

which coincides with (33).
b) Similarly, we have

$$tr W_\pm^\varphi(f) = \sqrt{2\pi} \int ds \tilde{f}(0, \pm\varphi(0)e^{-s}) = \int dvds f(v, \pm e^{-s})$$

as claimed. □

Definition 2.16. *By $\bar{\mathcal{B}}_\varphi$ we denote the Banach algebra obtained by equipping $\mathrm{L}^2(\mathbb{R}^2, d\mu)$ with the product and involution defined by (42) and (43) and extended from $\mathcal{B}'$ using (46) in the same manner as for the case $\varphi = 1$ treated previously.*

Theorem 2.17. *The involutive algebras $\mathcal{B}_\varphi$, resp. $\bar{\mathcal{B}}_\varphi$, where φ is positive and smooth, are isomorphic.*

Proof. For the case of $\mathcal{B}_\varphi$ we note that $\mathcal{F}^{-1}U\mathcal{F}$ maps $\mathcal{B}_\varphi$ onto $\mathcal{B} = \mathcal{B}_{\varphi=1}$ and is by construction a homomorphism by (23),(24),(40) and (41). The inverse map is obtained by replacing φ by η. The same argument applies to $\bar{\mathcal{B}}_\varphi$ since Theorem 2.15 shows that $\mathcal{F}^{-1}U\mathcal{F}$ is an isometry on $\mathcal{B}'$ and therefore its extension is an isometry from $\mathcal{B}_\varphi$ onto $\mathcal{B}$. $\qquad\square$

Remark 2.18. *The quantization maps $W_\pm$ were also considered in [1] and relations (33) and (34) were likewise derived.*
For the particular case $\varphi(a) = \frac{e^a - 1}{a}$, relations (46) and (47) also appear in [2].

3   Hopf algebra structure

The star algebras $\mathcal{B}_\varphi$ or $\bar{\mathcal{B}}_\varphi$ defined in the previous section obviously do not contain the coordinate functions α and β. Hence, to obtain a representation of M_κ with α and β representing the generators t and x in (14) we need an extension of the domain of definition for the star product and involution. It is the purpose of this section to exhibit such an extension.
As originally mentioned in [9] and developed in [10], M_κ has a natural Hopf algebra structure, which arises by dualization of the momentum subalgebra of the κ-Poincaré Hopf algebra. The coalgebra structure $(\triangle, \varepsilon)$ and antipode S are defined by

$$\triangle t = t \otimes 1 + 1 \otimes t, \quad \triangle x = x \otimes 1 + 1 \otimes x,$$
$$\varepsilon(t) = \varepsilon(x) = 0,$$
$$S(t) = -t, \quad S(x) = -x.$$

As will be seen in Theorem 3.6 below the extension we present allows a realization of the full Hopf algebra structure of κ-Minkowski space. Unless stated explicitly otherwise we work with the star product associated with the right invariant measure because of its simple form (25). Analogous results for the $\star$-product (44) are obtained similarly.

3.1   The algebra $\mathcal{C}$

Using standard notation $\partial_\alpha^n = \prod_{i=1}^k \frac{\partial}{\partial \alpha_i}$ for $\alpha = (\alpha_1, \ldots, \alpha_k) \in \mathbb{R}^k$, $n = (n_1, \ldots, n_k) \in \mathbb{N}_0^k$, $\mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$, and with $|\cdot|$ denoting the Euclidean norm on $\mathbb{R}^k$ we introduce the following function spaces.

Definition 3.1. *Let $\mathcal{C}_k$ be the space of smooth functions $f(\alpha, \beta)$ on $\mathbb{R}^{2k}$ satisfying polynomial bounds of the form*

$$|\partial_\alpha^n \partial_\beta^m f(\alpha, \beta)| \ \leq \ c_{n,m}(1 + |\alpha|)^{N_n}(1 + |\beta|)^{M_{n,m}}, \tag{48}$$

for all $\alpha, \beta \in \mathbb{R}^k$ and such that the Fourier transform $\tilde{f}$ of f (as a tempered distribution) w.r.t. α has compact support in α. Here, $n, m \in \mathbb{N}^k$ are arbitrary and $N_n, M_{n,m}$ are constants of which the former is independent of m, and $c_{n,m}$ is a positive constant.

Given $f \in C_k$ we denote by K_f the smallest compact subset of $\mathbb{R}^k$ such that $supp(f) \subseteq K_f \times \mathbb{R}^k$ and we call K_f the α-support of f.

For $k = 1$ we set $C = C_1$ and we have the canonical inclusion $C \otimes C \hookrightarrow C_2$ given by:

$$(f \otimes g)(\alpha_1, \alpha_2, \beta_1, \beta_2) = f(\alpha_1, \beta_1)g(\alpha_2, \beta_2),$$

for $f, g \in C$.

Remark 3.2. *Note that $\mathcal{B} \subseteq C$ and, additionally, if $f \in C$ and p is a polynomial in α, β, then p and pf are in C.*

In order to extend the $*$-product to C let $f, g \in C$ and define for fixed $\alpha, \beta \in \mathbb{R}$,

$$g_{\alpha,\beta}(v) = g(\alpha, e^{-v}\beta)e^{i\alpha v}, \quad v \in \mathbb{R}. \tag{49}$$

Motivated by (27) we then set

$$(f * g)(\alpha, \beta) = \frac{1}{\sqrt{2\pi}} \int dv\, \tilde{f}(v, \beta)g_{\alpha,\beta}(v) \tag{50}$$

which is well-defined since $\tilde{f}(v, \beta)$ has compact support in v and $g_{\alpha,\beta}$ is a smooth function. We show below that $f * g \in C$ and that

$$K_{f*g} \subseteq K_f + K_g. \tag{51}$$

In fact, viewing the product (27) as a linear map on $C \otimes C$, it extends to a linear map $m_* : C_2 \to C$ by setting

$$\begin{aligned}
(m_* F)(\alpha, \beta) &= \frac{1}{2\pi} \int d\alpha' \int dv\, \chi_F^1(v)F(\alpha', \alpha, \beta, e^{-v}\beta)e^{i(\alpha - \alpha')v} \\
&= \frac{1}{2\pi} \int d\alpha' \int dv\, \chi_F^1(v)F(\alpha + \alpha', \alpha, \beta, e^{-v}\beta)e^{-i\alpha' v}, \quad F \in C_2,
\end{aligned} \tag{52}$$

where χ_F^1 denotes a smooth function on $\mathbb{R}$ of compact support such that

$$\chi_F^1(v_1)\tilde{F}(v_1, v_2, \beta_1, \beta_2) = \tilde{F}(v_1, v_2, \beta_1, \beta_2)$$

as distributions, that is χ_F^1 equals 1 on a neighborhood of the projection of K_F on the first axis. Note that (52) coincides with (50) if $F = f \otimes g, f, g \in C$. Convergence of the double integral in (52) is a consequence of the polynomial bounds (48) for F, as can be seen as follows. Let ζ be a smooth function of compact support on $\mathbb{R}$ that equals 1 on a neighborhood of 0 and write the integral in (52) as a sum of two terms $F_1(\alpha, \beta)$ and $F_2(\alpha, \beta)$ where

$$F_1(\alpha, \beta) = \frac{1}{2\pi} \int d\alpha' \int dv\, \zeta(\alpha - \alpha')\chi_F(v)F(\alpha', \alpha, \beta, e^{-v}\beta)e^{i(\alpha - \alpha')v}. \tag{53}$$

Obviously, this latter integral is absolutely convergent and by repeated differentiation w.r.t. α, β it is seen that F_1 is smooth and satisfies polynomial bounds of the form (48). For F_2, given by formula (53) with ζ replaced by $1 - \zeta$, one obtains after N partial integrations w.r.t. v the expression

$$
\begin{aligned}
F_2(\alpha, \beta) &= \\
&= \frac{i^N}{2\pi} \int d\alpha' \int dv \, (\alpha - \alpha')^{-N} (1 - \zeta(\alpha - \alpha')) \frac{\partial^N}{\partial v^N} \left(\chi_F(v) F(\alpha', \alpha, \beta, e^{-v}\beta) \right) e^{i(\alpha - \alpha')v} \,.
\end{aligned}
\tag{54}
$$

Choosing N large enough one obtains an absolutely convergent integral as a consequence of (48), using that N_n is independent of m. Applying the same argument to derivatives of the integrand w.r.t. α, β it follows easily that F_2 is smooth and satisfies the bounds (48). In the Appendix it is proven that $m_* F$ is independent of the choice of χ_F^1 with the mentioned property and that

$$
\operatorname{supp}(m_* F) \subseteq \{ v_1 + v_2 \mid (v_1, v_2) \in K_F \} \times \mathbb{R} \,,
\tag{55}
$$

of which (51) is a special case. In particular, $m_*(F)$ has compact α-support and hence we may conclude that $m_*(F) \in \mathcal{C}$ for all $F \in \mathcal{C}_2$.

More generally, we can define maps $\mathcal{C}_{k+1} \to \mathcal{C}_k$ by letting m_* act on any two pairs of variables $(\alpha_i, \beta_i), (\alpha_j, \beta_j)$ while keeping the other variables fixed. We shall use the notation $m_* \otimes 1$ and $1 \otimes m_*$ for the maps $\mathcal{C}_3 \to \mathcal{C}_2$ where m_* acts on $(\alpha_1, \beta_1), (\alpha_2, \beta_2)$ and $(\alpha_2, \beta_2), (\alpha_3, \beta_3)$, respectively. Using arguments similar to those above one proves associativity of m_*, that is

$$
m_*(m_* \otimes 1) = m_*(1 \otimes m_*) \,.
\tag{56}
$$

Details are given in the appendix.

Next, we proceed to define the involution on $\mathcal{C}$. A convenient form is obtained from (28) which, after a simple change of variables, yields

$$
\int d\alpha d\beta \, f^*(\alpha, \beta) \, \tilde{\phi}(\alpha, \beta) = \int dv d\beta \, \bar{\tilde{f}}(v, \beta) \, \chi_f(v) \, \phi(-v, e^{-v}\beta) \, e^{-v} \,,
\tag{57}
$$

for $\phi \in \mathcal{S}(\mathbb{R}^2)$, where χ_f is an arbitrary smooth function of compact support that equals 1 on a neighborhood of K_f. Defining

$$
(R_f \phi)(v, \beta) = \chi_f(v) \, \phi(-v, e^{-v}\beta) \, e^{-v} \,,
\tag{58}
$$

for $\phi \in \mathcal{S}(\mathbb{R}^2)$, it is clear that R_f is a continuous mapping from $\mathcal{S}(\mathbb{R}^2)$ into itself. Hence, it follows that (57) defines f^* as a tempered distribution for any $f \in \mathcal{C}$. We refer to the appendix for a proof that f^* is independent of the choice of function χ_f with the asserted property. The Fourier transform of f^* w.r.t. α is given by

$$
\widetilde{f^*}(\phi) = \bar{\tilde{f}}(R_f \phi),
\tag{59}
$$

from which it is clear that the α-support of f^* fulfils

$$
K_{f^*} \subseteq -K_f \,.
\tag{60}
$$

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Hence we can choose $\chi_{f^*}(v) = \chi_f(-v)$. It is then easily verified that

$$R_f R_{f^*} \phi = \chi_f^2 \, \phi$$

which by use of (59) gives

$$\widetilde{f^{**}}(\phi) = \tilde{\bar{\bar{f}}}^*(R_{f^*}\phi) = \tilde{f}(R_f R_{f^*}\phi) = \tilde{f}(\phi)\,,$$

since χ_f^2 equals 1 on a neighbourhood of K_f. This shows that

$$f^{**} = f\,, \qquad f \in \mathcal{C}\,.$$

In order to show that $f^* \in \mathcal{C}$ for any $f \in \mathcal{C}$ we first note that f^* is, in fact, a function given by the following generalization of (26),

$$f^*(\alpha,\beta) = \frac{1}{2\pi} \int d\alpha' \int dv\, \chi_f(-v)\, \bar{f}(\alpha + \alpha', e^{-v}\beta)\, e^{-i\alpha'v}\,. \tag{61}$$

Indeed, convergence of this double integral is a consequence of (48), which can be seen by arguments similar to those for m_* as follows. Let again ζ be a smooth function on $\mathbb{R}$ of compact support that equals 1 on a neighborhood of 0 and write the integral in (61) as a sum of two terms $f_1^*(\alpha,\beta)$ and $f_2^*(\alpha,\beta)$, where

$$f_1^*(\alpha,\beta) = \frac{1}{2\pi} \int d\alpha' \int dv\, \zeta(\alpha')\chi_f(-v)\bar{f}(\alpha + \alpha', e^{-v}\beta)e^{-i\alpha'v}\,. \tag{62}$$

Clearly, this latter integral is absolutely convergent and by repeated differentiation w. r. t. α,β it follows easily that f_1^* is smooth and satisfies (48). For f_2^* one obtains after N partial integrations w. r. t. v the expression

$$f_2^*(\alpha,\beta) = \frac{(-i)^N}{2\pi} \int d\alpha' \int dv\, (1 - \zeta(\alpha'))\frac{\partial^N}{\partial v^N}\left(\chi_f(-v)\bar{f}(\alpha + \alpha', e^{-v}\beta)\right) e^{-i\alpha'v}\,. \tag{63}$$

By choosing N large enough this integral is absolutely convergent by (48). Applying the same argument to arbitrary derivatives of the integrand w. r. t. α,β it follows easily that f_2^* is smooth and satisfies (48). Having proven convergence of the integral (61), its coincidence with f^* follows easily. Hence, we have shown that f^* belongs to $\mathcal{C}$ and is given by (61).

Evidently, the preceding arguments can be extended to show that, more generally, an involution * on $\mathcal{C}_k$ is obtained by setting

$$F^*(\alpha,\beta) =$$
$$= (2\pi)^{-k} \int d^k\alpha' \int d^kv\chi_F(-v)\bar{F}(\alpha + \alpha', e^{-v_1}\beta_1, \ldots, e^{-v_k}\beta_k)e^{-i(\alpha_1'v_1 + \cdots + \alpha_k'v_k)}\,, \tag{64}$$

for $F \in \mathcal{C}_k$, where χ_F is a smooth function that equals 1 on a neighbourhood of K_F. We then have the following relation, whose proof is given in the appendix,

$$(m_*F)^* = m_*(F^*)^\wedge\,, \qquad F \in \mathcal{C}_2\,, \tag{65}$$

where $^\wedge$ denotes the flip operation on $\mathcal{C}_2$,

$$F^\wedge(\alpha_1, \alpha_2, \beta_1, \beta_2) = F((\alpha_2, \alpha_1, \beta_2, \beta_1)).$$

As a special case we get

$$(f * g)^* = g^* * f^* \tag{66}$$

for all $f, g \in \mathcal{C}$.

To summarize, we have established the following result.

Proposition 3.3. $\mathcal{C}$ *equipped with the $*$-product* (50) *and *-involution* (61) *is an involutive algebra.*

This algebra can be viewed as an involutive subalgebra of the multiplier algebra of $\mathcal{B}$:

Corollary 3.4. *If $f \in \mathcal{C}$ and $g \in \mathcal{B}$ then both $f * g$ and $g * f$ are in $\mathcal{B}$.*

Proof. It suffices to show the result only for $f * g$, since both $\mathcal{B}$ and $\mathcal{C}$ are involutive algebras. We know that $f * g$ is in $\mathcal{C}$ so it suffices to show that it is a Schwartz function whenever g is. Using (50) we have

$$\begin{aligned}
|f * g(\alpha, \beta)| &= \frac{1}{\sqrt{2\pi}} \left| \int d\alpha' f(\alpha', \beta) \mathcal{F}(\chi_f g_{\alpha,\beta})(\alpha') \right| \\
&\leq C(1 + |\beta|)^M \|\mathcal{F}(\chi_f g_{\alpha,\beta})\| \\
&\leq C'(1 + |\beta|)^M \|\chi_f g_{\alpha,\beta}\|',
\end{aligned}$$

where $\|\cdot\|, \|\cdot\|'$ are appropriate Schwartz norms, C, C' are constants and we have used (48). If g is a Schwartz function we clearly have

$$\|\chi_f g_{\alpha,\beta}\|' \leq C_{N',M'}(1 + |\alpha|)^{-N'}(1 + |\beta|)^{-M'}$$

for arbitrary $N', M' \geq 0$ and suitable constants $C_{N',M'}$. Hence $f * g$ is of rapid decrease. Similar arguments apply to derivatives of (50) w. r. t. α, β, thus proving that $f * g$ is a Schwartz function if g is. $\qquad\square$

Example 3.5. *It is useful to note the following instances of the $*$-product.*

   a)   *If $f, g \in \mathcal{C}$ where $g(\alpha, \beta) = g(\alpha)$ depends only on α then $f * g(\alpha, \beta) = f(\alpha, \beta)g(\alpha)$.*

   b)   *If $f, g \in \mathcal{C}$ and $f(\alpha, \beta) = f(\beta)$ depends only on β then $f * g(\alpha, \beta) = f(\beta)g(\alpha, \beta)$.*

   c)   *If $f(\alpha, \beta) = \alpha$ and $g \in \mathcal{C}$ depends only on β then*

$$(f * g)(\alpha, \beta) = \alpha g(\beta) + i\beta g'(\beta), \quad (g * f)(\alpha, \beta) = g(\beta)\alpha.$$

In particular, the constant function 1 is a unit of $\mathcal{C}$ and

$$\alpha * g(\beta) - g(\beta) * \alpha = i\beta g'(\beta).$$

For $g(\beta) = \beta$ this relation yields a representation of the defining relation (14) *in terms of a $*$-commutator with t, x corresponding to α, β. Note also that, $\alpha^* = \alpha$ and $\beta^* = \beta$ by* (61).

As a result we see that the star algebra $\mathcal{C}$ furnishes a representation of κ-Minkowski space, to be further developed in Theorem 3.6 below.

3.2 The Hopf algebra $\mathcal{C}$

We now proceed to discuss the coalgebra structure on $\mathcal{C}$ using the same notation for the coproduct, counit and antipode as for M_κ. The coproduct is of the standard cocommutative form

$$(\triangle f)(\alpha_1, \alpha_2, \beta_1, \beta_2) = f(\alpha_1 + \alpha_2, \beta_1 + \beta_2). \tag{67}$$

We show below that this defines a map $\triangle : \mathcal{C} \to \mathcal{C}_2$. The maps $\triangle \otimes 1$ and $1 \otimes \triangle$ have natural extensions to maps from $\mathcal{C}_2$ to $\mathcal{C}_3$, for which we shall use the same notation:

$$(\triangle \otimes 1)f(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3) = f(\alpha_1 + \alpha_2, \alpha_3, \beta_1 + \beta_2, \beta_3),$$
$$(1 \otimes \triangle)f(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3) = f(\alpha_1, \alpha_2 + \alpha_3, \beta_1, \beta_2 + \beta_3).$$

It is evident that $\triangle$ is coassociative, that is

$$(\triangle \otimes 1)\triangle = (1 \otimes \triangle)\triangle,$$

as maps from $\mathcal{C}_1$ to $\mathcal{C}_3$.

The counit $\varepsilon : \mathcal{C} \to \mathbb{C}$ and antipode $S : \mathcal{C} \to \mathcal{C}$ are defined by

$$\varepsilon(f) = f(0,0), \tag{68}$$

and

$$(Sf)(\alpha, \beta) = \overline{f^*}(-\alpha, -\beta), \tag{69}$$

respectively, for $f \in \mathcal{C}$. Obviously, ε and S are linear maps and it easily verified that

$$S^2 = \mathrm{Id}_\mathcal{C} .$$

We now state the main result of this section.

Theorem 3.6. *$\mathcal{C}$ equipped with the $*$-product, the *-involution, the coproduct $\triangle$, the counit ε and the antipode S defined above is a Hopf star algebra. Moreover, the algebra homomorphism ι from M_κ to $\mathcal{C}$ defined by*

$$\iota(t) = \alpha, \qquad \iota(x) = \beta$$

is compatible with the Hopf algebra structure:

$$\triangle \iota = (\iota \otimes \iota) \triangle, \qquad \varepsilon \iota = \varepsilon, \qquad S \iota = \iota S. \tag{70}$$

Proof. To show that $\mathcal{C}$ is a Hopf algebra we need to check that the coproduct and counit are well defined, that they are algebra homomorphisms and that S satisfies the conditions of an antipode, where the $*$-product on $\mathcal{C}_2$ is defined by a straightforward generalisation of (50) to functions of 4 variables, that is

$$(F * G)(\alpha_1, \alpha_2, \beta_1, \beta_2) = \frac{1}{(2\pi)^2} \int dv_1 \int dv_2 \int d\alpha'_1 \int d\alpha'_2$$
$$F(\alpha_1 + \alpha'_1, \alpha_2 + \alpha'_2, \beta_1, \beta_2) \, G(\alpha_1, \alpha_2, e^{-v_1}\beta_1, e^{-v_2}\beta_2) \, e^{-i(\alpha'_1 v_1 + \alpha'_2 v_2)}, \tag{71}$$

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for $F, G \in \mathcal{C}_2$.

That ε is a well-defined homomorphism follows by inserting $\alpha = \beta = 0$ in (50), which gives

$$\varepsilon(f * g) = \frac{1}{\sqrt{2\pi}} \int dv \, \tilde{f}(v, 0)g(0, 0) = f(0, 0) \, g(0, 0) = \varepsilon(f)\varepsilon(g) \, .$$

Concerning the coproduct it is evident that $\triangle f$ is a smooth function satisfying the polynomial bounds (48). Noting that

$$\widetilde{\triangle f}(v_1, v_2, \beta_1, \beta_2) = \sqrt{2\pi}\delta(v_1 - v_2)\tilde{f}(v_1, \beta_1 + \beta_2) \, , \tag{72}$$

where δ is the Dirac delta function, it follows that $\widetilde{\triangle f}$ has compact α-support with

$$K_{\triangle f} = \{(v, v) \mid v \in K_f\} \, .$$

Hence $\triangle f \in \mathcal{C}_2$.

To show that $\triangle$ is a homomorphism we write (71) as

$$(\triangle f) * (\triangle g)(\alpha_1, \alpha_2, \beta_1, \beta_2) =$$
$$= \frac{1}{2\pi} \int dv_1 dv_2 \, \widetilde{\triangle f}(v_1, v_2, \beta_1, \beta_2)\triangle g(\alpha_1, \alpha_2, e^{v_1}\beta_1, e^{v_2}\beta_2))e^{i(\alpha_1 v_1 + \alpha_2 v_2)} \, .$$

Using (72) this yields

$$(\triangle f) * (\triangle g)(\alpha_1, \alpha_2, \beta_1, \beta_2) = \frac{1}{\sqrt{2\pi}} \int dv \, \tilde{f}(v, \beta_1 + \beta_2)g(\alpha_1 + \alpha_2, e^{-v}(\beta_1 + \beta_2))e^{i(\alpha_1 + \alpha_2)v} \, ,$$

which is seen to be identical to $\triangle(f * g)(\alpha_1, \alpha_2, \beta_1, \beta_2)$ by (50) and (67), as desired.

For the antipode we need to demonstrate the relation

$$m_* \, (S \otimes 1)\triangle = \varepsilon \, \mathrm{Id}_{\mathcal{C}} = m_* \, (1 \otimes S)\triangle \, , \tag{73}$$

where $S \otimes 1$ and $1 \otimes S$ denote the natural extensions to $\mathcal{C}_2$ of the corresponding operators on $\mathcal{C} \otimes \mathcal{C}$. First, we use (61) and (67) to write

$$(S \otimes 1)\triangle f(\alpha_1, \alpha_2, \beta_1, \beta_2) = \frac{1}{2\pi} \int d\alpha' \int dv' \chi_f(v')f(\alpha', -e^{v'}\beta_1 + \beta_2)e^{i(\alpha_2 - \alpha_1 - \alpha')v'} \, .$$

Since this expression depends on (α_1, α_2) only through $\alpha_1 - \alpha_2$ it follows immediately from (52) that $m_*(S \otimes 1)\triangle f$ is independent of α. Hence we may set $\alpha = 0$ and obtain

$$m_*(S \otimes 1)\triangle f(\alpha, \beta) =$$
$$\frac{1}{(2\pi)^2} \int d\alpha'' \int dv'' \int d\alpha' \int dv' \, \chi_f(-v'')\chi_f(v')f(\alpha', -e^{v'}\beta + e^{-v''}\beta)e^{-i\alpha''(v' + v'')}e^{-i\alpha'v'} \, . \tag{74}$$

Let now ζ_1 be a smooth function on $\mathbb{R}$ of compact support which equals 1 on a neighborhood of 0, and define the functions ζ_R, $R > 0$, by $\zeta_R(v) = \zeta_1(\frac{v}{R})$, $v \in \mathbb{R}$. Then insert

$$1 = (\zeta_{R_1}(\alpha'') + (1 - \zeta_{R_1}(\alpha''))(\zeta_{R_2}(\alpha' + \alpha'') + (1 - \zeta_{R_2}(\alpha' + \alpha'')))$$

into the integrand and accordingly write the integral as a sum of four integrals by expanding the product on the right-hand side. By performing an adequate number of partial integrations w. r. t. v', v'' in the three terms containing at least one factor $(1 - \zeta_{R_1}(\alpha''))$ or $(1 - \zeta_{R_2}(\alpha' + \alpha''))$, we obtain absolutely convergent integrals that vanish in the limit $R_1, R_2 \to \infty$. In other words, the integral (74) can be obtained as the limit for $R_1, R_2 \to \infty$ of the absolutely convergent integrals defined by inserting an extra convergence factor $\zeta_{R_1}(\alpha'')\zeta_{R_2}(\alpha' + \alpha'')$. For the regularized integrals we then obtain, after integrating over α'', the expression

$$\frac{1}{(2\pi)^{3/2}} \int dv' dv'' d\alpha' \chi_f(-v'') \chi_f(v') R_1 \mathcal{F}(\zeta_1)(R_1(v' + v'')) \zeta_1(\frac{\alpha'}{R_2}) f(\alpha', -e^{v'}\beta + e^{-v''}\beta) e^{-i\alpha' v'}\,.$$

It is now easy to verify that the limit for $R_1 \to \infty$ equals

$$\frac{1}{2\pi} \int dv' d\alpha' \chi_f(v')^2 \, \zeta_1(\frac{\alpha'}{R_2}) f(\alpha', 0) \, e^{-i\alpha' v'}\,.$$

Finally, letting $R_2 \to \infty$ gives the result

$$\frac{1}{\sqrt{2\pi}} \int d\alpha' \mathcal{F}((\chi_f)^2)(\alpha') \, f(\alpha', 0) = f(0, 0)\,,$$

where we have used that $(\chi_f)^2$ equals 1 on a neighborhood of K_f. This proves the first equality in (73). The second one follows similarly.

So far, we have demonstrated that $\mathcal{C}$ is a Hopf algebra. To show that it is an involutive Hopf algebra it remains to verify that $\triangle$ and ε fulfil

$$(\triangle f)^* = \triangle(f^*) \qquad \text{and} \qquad \varepsilon(f^*) = \overline{\varepsilon(f)}\,, \tag{75}$$

where the involution on $\mathcal{C}_2$ is given by (64). The last relation in (75) is obvious. To establish the former we note that for $f \in \mathcal{C}$ and $\tau \in \mathcal{S}(\mathbb{R}^4)$ we get from (67) by a simple change of variables that the action of $\widetilde{\triangle f}$ as a distribution on τ is given by

$$\widetilde{\triangle f}(\tau) = \int d\alpha d\alpha' d\beta d\beta' \, f(\alpha, \beta) \tilde{\tau}(\alpha - \alpha', \alpha', \beta - \beta', \beta')\,.$$

Clearly, $\int d\alpha' d\beta' \tilde{\tau}(\alpha - \alpha', \alpha', \beta - \beta', \beta')$ is a Schwartz function of (α, β) whose Fourier transform w. r. t. α at (v, β) equals $\int d\beta' \tau(-v, -v, \beta - \beta', \beta')$. Thus, setting

$$(T\tau)(v, \beta) = \int d\beta' \tau(v, v, \beta - \beta', \beta')\,,$$

we have

$$\widetilde{\triangle f}(\tau) = \tilde{f}(T\tau)\,,$$

and hence by (59)

$$\widetilde{\triangle(f^*)}(\tau) = \widetilde{f^*}(T\tau) = \bar{\tilde{f}}(R_f T\tau)\,.$$

Using (58) we have

$$\chi_f(v)(R_f T\tau)(v,\beta) = \chi_f(v)^2 \int d\beta'\, \tau(-v,-v,e^{-v}\beta - \beta',\beta')\, e^{-v}$$

$$= \chi_f(v)^2 \int d\beta'\, \tau(-v,-v,e^{-v}(\beta - \beta'),e^{-v}\beta')\, e^{-2v}$$

$$= T(R_f \otimes R_f)\tau(v,\beta)\,.$$

Inserting this into the previous equation yields

$$\widetilde{\triangle(f^*)}(\tau) = \bar{\bar{f}}(T(R_f \otimes R_f)\tau) = \overline{\widetilde{\triangle f}}((R_f \otimes R_f)\tau) = \widetilde{(\triangle f)^*}(\tau)\,,$$

which proves the claim.

Finally, knowing that $\triangle, \varepsilon$ are homomorphisms and S an antihomomorphism, the compatibility relations (70) follow by verifying their validity for the generators t, x. For $\triangle$ and ε this is trivial to verify. As noted in Example 3.5 we have $\alpha^* = \alpha$ and $\beta^* = \beta$ such that

$$S(\alpha) = -\alpha \qquad \text{and} \qquad S(\beta) = -\beta\,.$$

On the other hand, the antipode on M_κ fulfils $St = -t\,, Sx = -x$, which shows that the relation for S in (70) also holds for the generators. $\qquad\square$

Remark 3.7. *The homomorphism property of $\triangle$ proven above holds more generally in the form*

$$M_*(\triangle \otimes \triangle)F = \triangle(m_*F)\,, \quad F \in \mathcal{C}_2\,, \tag{76}$$

where $M_* : \mathcal{C}_4 \to \mathcal{C}_2$ denotes the canonical extension of (71) given by

$$M_*H(\alpha_1,\alpha_2,\beta_1,\beta_2) = \frac{1}{(2\pi)^2} \int dv_1 \int dv_2 \int d\alpha_1' \int d\alpha_2'$$

$$\chi_H^1(v_1)\chi_G^2(v_2)\, H(\alpha_1+\alpha_1', \alpha_2+\alpha_2', \alpha_1, \alpha_2, \beta_1, \beta_2, e^{-v_1}\beta_1, e^{-v_2}\beta_2)\, e^{-i(\alpha_1'v_1+\alpha_2'v_2)}\,, \tag{77}$$

for $H \in \mathcal{C}_4$, where χ_H^1, χ_H^2 denote smooth functions of compact support that equal 1 on a neighborhood of the projection of K_H onto the first and second axis, respectively. The verification of (77) is left to the reader.

Remark 3.8. *We have above used the $*$-product and *-involution associated with the right invariant Haar measure on G to equip $\mathcal{C}$ with a Hopf algebra structure compatible with that of κ-Minkowski space. The reader may easily check that by similar arguments one obtains an alternative realization of M_κ on the basis of the $\star$-product and *-involution associated with the left invariant Haar measure.*

4 Lorentz covariance

A salient feature of κ-Minkowski space is the existence of an action on it of a deformation of the Poincaré Lie algebra [9, 10], the so-called κ-Poincaré algebra. In two dimensions, the latter

is usually presented as the Hopf algebra with generators E, P and N, the energy, momentum and Lorentz boost, respectively, fulfilling the relations

$$
\begin{aligned}
&[P, E] = 0, \quad [N, E] = P, \\
&\triangle E = E \otimes 1 + 1 \otimes E, \quad \triangle P = P \otimes 1 + e^{-E} \otimes P, \\
&[N, P] = \frac{1}{2}(1 - e^{-2E}) - \frac{1}{2}P^2, \quad \triangle N = N \otimes 1 + e^{-E} \otimes N,
\end{aligned}
\tag{78}
$$

and with counit annihilating the generators whereas the antipode acts according to

$$
S(E) = -E, \quad S(P) = -e^E P, \quad S(N) = -e^E N.
$$

As mentioned previously, the algebraic κ-Minkowski space M_κ can be defined as the dual of the Hopf subalgebra generated by E and P [10]. It is the purpose of this section to exhibit explicitly the action of the κ-Poincaré algebra in terms of linear operators on the realization $\mathcal{C}$ of M_κ developed in the previous section. Moreover, we shall find that by restriction we obtain an action of the κ-Poincaré algebra on the smaller algebra $\mathcal{B}$.

To avoid the appearance of the exponential of E in (78) we prefer to introduce it as an invertible generator $\mathcal{E}$ and define the κ-Poincaré algebra $\mathcal{P}_\kappa$ accordingly as the Hopf algebra generated by $E, P, \mathcal{E}, N$ fulfilling

$$
\begin{aligned}
&[P, E] = [P, \mathcal{E}] = [E, \mathcal{E}] = 0, \\
&[N, E] = P, \quad [N, \mathcal{E}] = -\mathcal{E}P, \quad [N, P] = \frac{1}{2}(1 - \mathcal{E}^2) - \frac{1}{2}P^2, \\
&\triangle E = E \otimes 1 + 1 \otimes E, \quad \triangle P = P \otimes 1 + \mathcal{E} \otimes P, \\
&\triangle \mathcal{E} = \mathcal{E} \otimes \mathcal{E}, \quad \triangle N = N \otimes 1 + \mathcal{E} \otimes N,
\end{aligned}
\tag{79}
$$

and with counit and antipode given by

$$
\varepsilon(E) = \varepsilon(P) = \varepsilon(N) = 0, \quad \varepsilon(\mathcal{E}) = 1,
\tag{80}
$$

$$
S(E) = -E, \quad S(\mathcal{E}) = \mathcal{E}^{-1}, \quad S(P) = -\mathcal{E}^{-1}P, \quad S(N) = -\mathcal{E}^{-1}N.
\tag{81}
$$

We also observe that, although the κ-Poincaré algebra was originally introduced without involution, it is easy to verify that

$$
E^* = E, \quad P^* = P, \quad N^* = -N, \quad \mathcal{E}^* = \mathcal{E},
\tag{82}
$$

defines an involution on $\mathcal{P}_\kappa$ making it a Hopf star algebra. Note, however, that the involution does not commute with S.

4.1 Action of the momentum subalgebra on $\mathcal{C}$

In order to define the action of $P, E, \mathcal{E}$ on $\mathcal{C}$ we first make a slight digression on imaginary translations of elements in $\mathcal{C}$.

Let $f \in \mathcal{C}$. Since $\tilde{f}$ has compact α-support it follows (see e.g. [13]) that f can be analytically continued to an entire function of α. The analytic continuation will likewise be denoted by f and is given by

$$f(\alpha + i\gamma, \beta) = \frac{1}{\sqrt{2\pi}} \tilde{f}(\chi_f(v)e^{i(\alpha+i\gamma)v}, \beta) = \frac{1}{\sqrt{2\pi}} \int d\alpha' f(\alpha + \alpha', \beta) \mathcal{F}(e^{-\gamma v}\chi_f(v))(\alpha') . \quad (83)$$

For fixed $\gamma \in \mathbb{R}$ we claim that the function $T_\gamma f$ defined by

$$(T_\gamma f)(\alpha, \beta) = f(\alpha + i\gamma, \beta) \quad (84)$$

belongs to $\mathcal{C}$. Indeed, since $\mathcal{F}(e^{-\gamma v}\chi_f(v))(\alpha')$ is a Schwartz function of α' we get immediately from (83) that the derivatives of $T_\gamma f$ are obtained by differentiating the integrand and, combining this with (48), it follows easily that $T_\gamma f$ fulfils polynomial bounds of the form (48). That $\widetilde{T_\gamma f}$ has compact α-support follows from

$$\widetilde{T_\gamma f}(v, \beta) = e^{-\gamma v}\tilde{f}(v, \beta) . \quad (85)$$

From this relation or, alternatively, from the uniqueness of the analytic continuation of f in α we conclude that the *imaginary translation operators* $T_\gamma : \mathcal{C} \to \mathcal{C}$ form a one-parameter group,

$$T_{\gamma+\eta} = T_\gamma T_\eta , \quad \gamma, \eta \in \mathbb{R} , \qquad T_0 = \mathrm{Id}_\mathcal{C} .$$

Similarly, n-parameter groups of imaginary translation operators $T_{\underline{\gamma}}$ are defined on $\mathcal{C}_n$ for any $n \in \mathbb{N}$ and $\underline{\gamma} \in \mathbb{R}^n$. We shall write T_γ for $T_{(\gamma,\dots,\gamma)}$, independently of n, for $\gamma \in \mathbb{R}$.
We next note the following two properties of these maps.

Proposition 4.1. *For fixed $\gamma \in \mathbb{R}$ the map $T_\gamma : \mathcal{C} \to \mathcal{C}$ is an algebra automorphism, that is*

$$T_\gamma(m_* F) = m_*(T_\gamma F) , \quad F \in \mathcal{C}_2 . \quad (86)$$

Moreover,

$$T_\gamma(f^*) = (T_{-\gamma}f)^* , \qquad f \in \mathcal{C}. \quad (87)$$

Proof. Since $T_\gamma^{-1} = T_{-\gamma}$, it is sufficient to verify (86) and (87). By (52) we have

$$m_* T_\gamma F(\alpha, \beta) = \int d\alpha' \int dv \chi_F^1(v) F(\alpha' + \alpha + i\gamma, \alpha + i\gamma, \beta, e^{-v}\beta) e^{-i\alpha'v} . \quad (88)$$

That this is an entire function of $z = \alpha + i\gamma$ for fixed β is seen as follows. By inserting a convergence factor $\zeta_R(\alpha')$ into the integrand we have, as seen previously, that the regularized integrals converge to the integral (88) as $R \to \infty$ for fixed z. It is easy to see that the convergence is uniform in z on compact subsets of $\mathbb{C}$. Since the regularized integrals are obviously analytic in z it follows that the same holds for (88). Hence this is the unique entire function whose restriction to $\mathbb{R}$ coincides with $m_* F(\alpha, \beta)$ for fixed β. But this function is by definition equal to the lefthand side of (86) for $z = \alpha + i\gamma$. This concludes the proof of (86).

Concerning (87) we note that by (61)

$$(T_{-\gamma}f)^*(\alpha, \beta) = \frac{1}{2\pi} \int d\alpha' \int dv \, \chi_f(-v) \bar{f}(\alpha - i\gamma + \alpha', e^{-v}\beta) e^{-i\alpha'v}, \qquad (89)$$

which by similar arguments as those above is seen to be an entire function of $z = \alpha + i\gamma$ for fixed β. Since it coincides with $f^*(\alpha, \beta)$ for $\gamma = 0$ we conclude that it equals the lefthand side of (87) for all $z \in \mathbb{C}$. This proves (87).

$\square$

By the preceding analyticity argument we obtain

$$T_{\gamma}(f^*)(\alpha, \beta) = \frac{1}{2\pi} \int d\alpha' \int dv \, \chi_f(-v) \bar{f}(\alpha', e^{-v}\beta) e^{-\gamma v} e^{i(\alpha - \alpha')v}, \qquad (90)$$

for $f \in \mathcal{C}$, since the right hand side is seen to be an analytic function of $z = \alpha + i\beta$ that coincides with the right hand side of (89) for $\gamma = 0$.

Now we can state the main result of this subsection on the action of the Hopf subalgebra generated by $E, P, \mathcal{E}$, called the *extended momentum algebra*, on $\mathcal{C}$.

Theorem 4.2. *The algebra $\mathcal{C}$ is an involutive Hopf module algebra with respect to the following linear action of the extended momentum algebra on $\mathcal{C}$:*

$$E \triangleright f = -i\frac{\partial f}{\partial \alpha}, \quad P \triangleright f = -i\frac{\partial f}{\partial \beta}, \quad \mathcal{E} \triangleright f = T_1 f. \qquad (91)$$

Proof. It is clear that the actions of $E, P, \mathcal{E}$ defined by (91) are linear on $\mathcal{C}$ and are mutually commuting. Therefore, it only remains to verify the compatibility of the action with the $*$-product and involution. That

$$E \triangleright (m_* F) = m_*(E \otimes 1) \triangleright F + m_*(1 \otimes E) \triangleright F$$

is obvious from (52) since differentiation w.r.t. α in the integrand is permitted by a standard convergence argument. For the action of $\mathcal{E}$ we have that

$$\mathcal{E} \triangleright (m_* F) = m_*(\mathcal{E} \otimes \mathcal{E}) \triangleright F,$$

which is a special case of (86). Finally, for the action of P we have

$$(P \triangleright (m_* F))(\alpha, \beta) = \frac{-i}{(2\pi)^2} \int d\alpha' \int dv \, \chi_F^1(v) \left(\frac{\partial F}{\partial \beta_1}(\alpha', \alpha, \beta, e^{-v}\beta) + e^{-v} \frac{\partial F}{\partial \beta_2}(\alpha', \alpha, \beta, e^{-v}\beta) \right) e^{i(\alpha - \alpha')v}, \qquad (92)$$

where it is seen that the contribution from the first term in parenthesis evidently equals $m_*(P \otimes 1)F(\alpha, \beta)$.

On the other hand, from (83) we get

$$((\mathcal{E} \otimes 1) \triangleright F)(\alpha_1, \alpha_2, \beta_1, \beta_2) = \frac{1}{2\pi} \int d\alpha'_1 \int dv_1 \, \chi_F^1(v_1) F(\alpha_1 + \alpha'_1, \alpha_2, \beta_1, \beta_2) e^{-v_1} e^{-i\alpha'_1 v_1}$$

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and hence

$$m_* \left((\mathcal{E} \otimes 1) \triangleright F \right) (\alpha, \beta) = \frac{1}{2\pi} \int d\alpha'_2 \int dv_2 \int d\alpha'_1 \int dv_1$$
$$\chi_F^1(v_1) \chi_F^1(v_2)\, F(\alpha + \alpha'_1 + \alpha'_2, \alpha, \beta, e^{-v_2}\beta) e^{-v_1} e^{-i\alpha'_1 v_1 + \alpha'_2 v_2}\ .$$

By introducing convergence factors $\zeta_{R_1}(\alpha'') \zeta_{R_2}(\alpha' + \alpha'')$ as in the proof of (73) above we obtain after integrating over α'' and taking the limit $R_1, R_2 \to \infty$ that

$$m_* \left((\mathcal{E} \otimes 1) \triangleright F \right) (\alpha, \beta) = \int d\alpha' \int dv' \chi_F^1(v')^2\, F(\alpha + \alpha', \alpha, \beta, e^{-v'}\beta) e^{-v} e^{-i\alpha'v'}\ .$$

Using that $(\chi_F^1)^2$ equals 1 on a neighborhood of the projection of K_F onto the first axis we see that the second term in parenthesis in (92) yields the contribution $m_*(\mathcal{E} \otimes P) F(\alpha, \beta)$. Hence, we have shown that

$$P \triangleright (m_* F) = m_*(P \otimes 1) \triangleright F + m_*(\mathcal{E} \otimes P) \triangleright F, \quad F \in \mathcal{C}_2\,, \tag{93}$$

which concludes the argument that the action of the extended momentum algebra is compatible with multiplication on $\mathcal{C}$.

Compatibility of the action with the involution is the statement that

$$(h \triangleright f)^* = (Sh)^* \triangleright f^*\,, \tag{94}$$

for $f \in \mathcal{C}$ and h in the extended momentum algebra. It suffices to verify this for the generators $E, P, \mathcal{E}$. For $\mathcal{E}$ it follows directly from (87), whereas for E it is a consequence of (61) by differentiating both sides with respect to α.

Differentiating (61) with respect to β and using (90) we obtain

$$P \triangleright f^* = -\mathcal{E}(P \triangleright f)^*\,,$$

which gives

$$(P \triangleright f)^* = -(\mathcal{E}^{-1}P) \triangleright f^*\,.$$

Since $S(P)^* = (-\mathcal{E}^{-1}P)^* = -\mathcal{E}^{-1}P$ it follows that (94) is satisfied for $h = P$. This completes the proof of the theorem. $\qquad\square$

4.2 Action of $\mathcal{P}_\kappa$ on $\mathcal{C}$

To represent the boost operator N by a linear action on $\mathcal{C}$ we introduce the operators of multiplication by α and β as

$$(L_\alpha f)(\alpha, \beta) = \alpha f(\alpha, \beta), \quad (L_\beta f)(\alpha, \beta) = \beta f(\alpha, \beta),$$

for $f \in \mathcal{C}$.

Lemma 4.3. L_α and L_β are linear operators on $\mathcal{C}$ which satisfy the following rules with respect to the product and involution on $\mathcal{C}$:

$$L_\alpha(m_*F) = m_*(1 \otimes L_\alpha)F = m_*(L_\alpha \otimes 1)F + m_*(1 \otimes L_\beta P)F\,, \qquad (95)$$

$$L_\beta(m_*F) = m_*(L_\beta \otimes 1)F = m_*(\mathcal{E}^{-1} \otimes L_\beta)F\,, \qquad (96)$$

$$(L_\alpha f)^* = L_\alpha f^* - L_\beta P f^* \quad \text{and} \quad (L_\beta f)^* = \mathcal{E} L_\beta f^*\,. \qquad (97)$$

Proof. The two left identities in (95) and (96) follow immediately from (52). From the last expression in (52) we obtain

$$L_\alpha(m_*F)(\alpha, \beta) = m_*(L_\alpha \otimes 1)F(\alpha, \beta) - \int d\alpha' \int dv \chi_F^1(v)\alpha' F(\alpha + \alpha', \alpha, \beta, e^{-v}\beta)e^{-i\alpha v}$$

$$= m_*(L_\alpha \otimes 1)F(\alpha, \beta) + m_*(1 \otimes L_\beta P)F(\alpha, \beta)\,,$$

where the last step follows by a partial integration w. r. t. v. This proves the second identity in (95). Similarly, the second identity in (96) is obtained from

$$L_\beta(m_*F)(\alpha, \beta) = \int d\alpha' \int dv \chi_F^1(v)e^v F(\alpha + \alpha', \alpha, \beta, e^{-v}\beta)e^{-v}\beta e^{-i\alpha'v}$$

$$= m_*(\mathcal{E}^{-1} \otimes L_\beta)F(\alpha, \beta)\,,$$

where the last step follows by the same argument as in the proof of (93) above.

The second identity of (97) follows immediately from (61) and (90). For the first one we multiply both sides of (61) by α and obtain after a partial integration

$$L_\alpha f^*(\alpha, \beta) = (L_\alpha f)^*(\alpha, \beta) - (L_\beta P f)^*(\alpha, \beta)\,.$$

Using the the second identity of (97) and (94) for $h = P$ the first identity of (97) follows. $\square$

We are now in a position to extend Theorem 4.2 as follows.

Theorem 4.4. *Defining the linear action of N on $\mathcal{C}$ by*

$$N = -iL_\alpha P - \frac{i}{2}(1 - \mathcal{E}^2)L_\beta + \frac{i}{2}L_\beta P^2\,, \qquad (98)$$

and the action of $E, P, \mathcal{E}$ as in (91) then $\mathcal{C}$ becomes an involutive Hopf module algebra of $\mathcal{P}_\kappa$.

Proof. That N, P, E and $\mathcal{E}$ satisfy the commutation relations of (79) is easily seen by inspection. It remains to check that the action of N on $\mathcal{C}$ is compatible with the product and involution on $\mathcal{C}$ using the coproduct of (79). By (93) and Lemma 4.3 one gets, for $F \in \mathcal{C}$,

$$N \triangleright m_*F = \left(-iL_\alpha P - \frac{i}{2}(1 - \mathcal{E}^2)L_\beta + \frac{i}{2}L_\beta P^2 \right) m_*F$$

$$= - iL_\alpha\, m_*(P \otimes 1)F - iL_\alpha m_*(\mathcal{E} \otimes P)F$$

$$- \frac{i}{2}L_\beta\, m_*F + \frac{i}{2}\mathcal{E}^2 m_*(L_\beta \otimes 1)F$$

$$+ \frac{i}{2}L_\beta\, m_* \left((P^2 \otimes 1)F + 2(\mathcal{E}P \otimes P)F + (\mathcal{E}^2 \otimes P^2)F \right)\,.$$

Making further use of Lemma 4.3 and (86) this expression equals

$$- im_*(L_\alpha P \otimes 1)F - im_*(P \otimes L_\beta P)F) - im_*(\mathcal{E} \otimes L_\alpha P)F$$
$$- \frac{i}{2}m_*(L_\beta \otimes 1)F + \frac{i}{2}m_*(\mathcal{E}^2 L_\beta \otimes \mathcal{E}^2)F + \frac{i}{2}m_*(L_\beta P^2 \otimes 1)F$$
$$+ im_*(P \otimes L_\beta P)F + \frac{i}{2}m_*(\mathcal{E} \otimes L_\beta P^2)F \,.$$

Here, two terms are seen to cancel, and using the relation

$$m_*(\mathcal{E}^2 L_\beta \otimes 1)F - m_*(\mathcal{E} \otimes L_\beta)F = 0 \,,$$

which follows from (96), we can rewrite the last expression in the form

$$m_*\left(-i(L_\alpha P \otimes 1)F - \frac{i}{2}(L_\beta \otimes 1)F + \frac{i}{2}(\mathcal{E}^2 L_\beta \otimes 1)F + \frac{i}{2}(L_\beta P^2 \otimes 1)F)\right)$$
$$+ m_*\left(-i(\mathcal{E} \otimes L_\alpha P)F - \frac{i}{2}(\mathcal{E} \otimes L_\beta)F + \frac{i}{2}(\mathcal{E} \otimes \mathcal{E}^2 L_\beta)F + \frac{i}{2}(\mathcal{E} \otimes L_\beta P^2)F\right))$$
$$= m_*((N \otimes 1) \triangleright F + (\mathcal{E} \otimes N) \triangleright F) \,.$$

This proves compatibility of the action of N with the product on $\mathcal{C}$..
Using (94) for $h = P$ and $h = \mathcal{E}$ and (97) we get

$$(N \triangleright f)^* = \left(\left(-iL_\alpha P - \frac{i}{2}(1 - \mathcal{E}^2)L_\beta + \frac{i}{2}L_\beta P^2\right)f\right)^*$$
$$= \left(-iL_\alpha \mathcal{E}^{-1} P + iL_\beta P\mathcal{E}^{-1}P + \frac{i}{2}(1 - \mathcal{E}^{-2})L_\beta \mathcal{E} - \frac{i}{2}L_\beta \mathcal{E}\mathcal{E}^{-2}P^2\right)f^*$$
$$= \left(-iL_\alpha P - \frac{i}{2}(1 - \mathcal{E}^2)L_\beta + \frac{i}{2}L_\beta P^2\right)\mathcal{E}^{-1}f^*$$
$$= N\mathcal{E}^{-1} \triangleright f^*,$$

Noting that $(S(N))^* = -(\mathcal{E}^{-1}N)^* = N\mathcal{E}^{-1}$, this proves compatibility of the action of N with involution. □

In view of the obvious fact that $\frac{\partial}{\partial \alpha}, \frac{\partial}{\partial \beta}, L_\alpha, L_\beta$ and T_1 all map $\mathcal{B}$ into itself, the following is a consequence of Theorem 4.4.

Corollary 4.5. *The subalgebra $\mathcal{B}$ of $\mathcal{C}$ is an involutive Hopf module algebra for $\mathcal{P}_\kappa$ with action defined by* (91) *and* (98).

Remark 4.6. *By inspection of* (79), (80), (81) *it is seen that setting*

$$\Lambda_q(E) = E \,, \quad \Lambda_q(P) = P \,, \quad \Lambda_q(\mathcal{E}) = \mathcal{E} \,, \quad \Lambda_q(N) = N + qP \,,$$

defines a Hopf algebra automorphism Λ_q of $\mathcal{P}_\kappa$ for each $q \in \mathbb{C}$. As a consequence, one obtains an involution on $\mathcal{P}_\kappa$ for any $q \in \mathbb{R}$ by replacing $N^* = -N$ in (82) *by*

$$N^* = -N + qP \,. \tag{99}$$

For this involution Theorem 4.4 is still valid if N as given by (98) is replaced by

$$N' = N + \frac{q}{2}P.$$

The particular choice $q = 1$ ensures that the operator N' is antisymmetric w. r. t. the L^2-inner product on $\mathcal{B}$, as is easily verified. More generally, it follows that the action of h^* on $\mathcal{B}$ in this case coincides with that of the adjoint of h w. r. t. the L^2-inner product on $\mathcal{B}$, for any $h \in \mathcal{P}_\kappa$, see Proposition 4.7 below.

On $\mathcal{B}$ the integral w. r. t. $d\alpha d\beta$ is well defined as a linear form that we shall denote by $\int$. In the following proposition we collect some basic properties of $\int$ in relation to the module algebra structure on $\mathcal{B}$.

Proposition 4.7. *a) The integral w. r. t. the uniform measure on $\mathbb{R}^2$ is invariant under the action of $\mathcal{P}_\kappa$ on $\mathcal{B}$ defined above in the sense that, for any $h \in \mathcal{P}_\kappa$ and $f \in \mathcal{B}$,*

$$\int h \triangleright f = \varepsilon(h) \int f.\tag{100}$$

b) $\int$ is a left and right invariant integral on the Hopf algebra $\mathcal{B}$ in the sense that

$$\left(\int \otimes \, \mathrm{Id} \right) \triangle f = \int f = \left(\mathrm{Id} \otimes \int \right) \triangle f, \quad f \in \mathcal{B}.\tag{101}$$

c)

$$\int Sf = \int f \quad and \quad \int f * (Sg) = \int g * (Sf).\tag{102}$$

d) For any $f, g \in \mathcal{B}$ and $h \in \mathcal{C}$ we have

$$\int (h \triangleright f) * g^* = \int f * (h^* \triangleright g)^*,\tag{103}$$

if the involution on $\mathcal{P}_\kappa$ is defined by (99) for $q = 1$ and the action of $E, P, \mathcal{E}, N$ on $\mathcal{B}$ are given by (91) and

$$N \triangleright f = (-iL_\alpha P - \frac{i}{2}(1 - \mathcal{E}^2)L_\beta + \frac{i}{2}PL_\beta P)f, \quad f \in \mathcal{B}.\tag{104}$$

e) For any $f, g \in \mathcal{B}$ we have

$$\int f * g = \int (\mathcal{E} \triangleright g) * f.\tag{105}$$

which means that $\int$ is a twisted trace.

Proof. a) It suffices to verify (100) for the generators $E, P, \mathcal{E}$ and N. First, since both E and P act on f as partial derivatives

$$\int d\alpha d\beta \, (P \triangleright f)(\alpha, \beta) = 0 = \int d\alpha d\beta \, (E \triangleright f)(\alpha, \beta).\tag{106}$$

For $\mathcal{E}$ we have

$$\int d\alpha d\beta \, (\mathcal{E} \triangleright f)(\alpha, \beta) = \int d\alpha d\beta \, f(\alpha + i, \beta) = \int d\alpha d\beta \, f(\alpha, \beta)$$

as a consequence of Cauchy's theorem. Finally, for the action of N, one uses the identities

$$L_\beta P^2 = P^2 L_\beta - 2P, \quad L_\alpha P = P L_\alpha,$$

to deduce from the preceding results that

$$\int d\alpha d\beta \, (N \triangleright f)(\alpha, \beta) = 0.$$

This finishes the proof of a).

b) Identities (101) follow trivially from the translation invariance of the measure $d\alpha d\beta$.

c) The first identity of (102) follows from (36) and (69):

$$\int d\alpha d\beta \, (Sf)(\alpha, \beta) = \int d\alpha d\beta \, (Sf)(-\alpha, -\beta) = \int d\alpha d\beta \, \overline{f^*}(\alpha, \beta) = \int d\alpha d\beta \, f(\alpha, \beta) \, .$$

The second identity follows from the former by using that S is an antihomomorphism and $S^2 = \mathrm{Id}$ on $\mathcal{B}$.

d) By (35) we see that (103) is equivalent to the statement that the action of $h^* \in \mathcal{P}_\kappa$ on $\mathcal{B}$ as a linear operator on $\mathcal{B} \subset L^2(\mathbb{R}^2)$ equals the action of the adjoint of h w. r. t. the standard inner product on $L^2(\mathbb{R}^2)$. That this holds for E and P is clear form (91). For $\mathcal{E}$ we have

$$\int (\mathcal{E} \triangleright f) * g^* = \int \mathcal{E} \triangleright (f * (\mathcal{E}^{-1} \triangleright g^*)) = \int f * (\mathcal{E} \triangleright g)^* \, ,$$

by (86), (87) and (100). This proves (103) for $h = \mathcal{E}$. Since L_α and L_β are symmetric operators on $\mathcal{B} \subset L^2(\mathbb{R}^2)$ one can now check by direct computation that N as given by (98) is antisymmetric.

e) Using Cauchy's theorem and a change of variables we get from (25) that

$$\begin{aligned}
\int (\mathcal{E} \triangleright g) * f &= \frac{1}{2\pi} \int d\alpha d\beta \int dv \int d\alpha' \, g(\alpha + \alpha' + i, \beta) f(\alpha, e^{-v}\beta) e^{-i\alpha' v} \\
&= \frac{1}{2\pi} \int d\alpha d\beta \int dv \int d\alpha' \, g(\alpha', \beta) f(\alpha, e^{-v}\beta) \, e^{-v} e^{i(\alpha - \alpha')v} \\
&= \int d\beta dv \, \tilde{g}(v, \beta) \tilde{f}(-v, e^{-v}\beta) \, e^{-v} \, .
\end{aligned}$$

A change of variables shows that the last expression equals $\int d\beta dv \, \tilde{f}(v, \beta) \tilde{g}(-v, e^{-v}\beta)$ which by reversing the steps above yields $\int f * g$. This completes the proof. $\qquad \square$

4.3 Explicit dependence on the kappa parameter

For the sake of completeness we end this section by reintroducing the κ-parameter which we eliminated at the outset by rescaling the t generator of M_κ. The correct dependence on κ for both M_κ and $\mathcal{P}_\kappa$ is obtained by simply rescaling the variables α, β by κ, i.e. set $(\alpha, \beta) = (\kappa\hat{\alpha}, \kappa\hat{\beta})$ and express the (co)algebra operations in terms of the dimensionful variables $\hat{\alpha}, \hat{\beta}$, and then rename the latter (α, β). Explicitly, the $*$-product on $\mathcal{B}$ is replaced by

$$f *_\kappa g(\alpha, \beta) = \frac{1}{2\pi} \int d\alpha' dv\, f(\alpha + \alpha', \beta) g(\alpha, e^{-\frac{v}{\kappa}}\beta) e^{-i\alpha v}, \tag{107}$$

and the involution is changed to

$$f^*(\alpha, \beta) = \frac{1}{2\pi} \int d\alpha' dv\, \bar{f}(\alpha + \alpha', e^{-\frac{v}{\kappa}}\beta) e^{-i\alpha' v}, \tag{108}$$

whereas the coproduct and counit are unchanged. Furthermore, the action of the operators $E, P, \mathcal{E}, N$ on $\mathcal{B}$ are redefined as

$$E \triangleright f = -i\frac{\partial f}{\partial \alpha}, \quad P \triangleright f = -i\frac{\partial f}{\partial \beta}, \quad \mathcal{E} \triangleright f = T_{\frac{1}{\kappa}} f,$$

$$N = -iL_\alpha P - \frac{i\kappa}{2}\left(1 - \mathcal{E}^2\right) L_\beta + \frac{i}{2\kappa} L_\beta P^2,$$

where L_α, L_β denote multiplication by α, β, respectively, as before. With these definitions we obtain a function algebra realization $\mathcal{B}$ of M_κ and a representation of the involutive Hopf algebra $\mathcal{P}_\kappa$ on $\mathcal{B}$, as displayed in e.g. [10].

Finally, we note the following series representation of the $*_\kappa$-product for sufficiently regular functions. For simplicity we consider a rather restricted class of functions but the proof can be adapted to more general situations.

Proposition 4.8. *If $f, g \in \mathcal{B}$ and $g(\alpha, \beta)$ is an entire function of β then*

$$(f *_\kappa g)(\alpha, \beta) = \sum_{n=0}^{\infty} \frac{i^n}{\kappa^n n!} \partial_\alpha^n f(\alpha, \beta) (\beta\partial_\beta)^n g(\alpha, \beta),$$

for all $(\alpha, \beta) \in \mathbb{R}^2$.

Proof. First rewrite (107) as

$$(f *_\kappa g)(\alpha, \beta) = \frac{1}{\sqrt{2\pi}} \int dv\, \tilde{f}(v, \beta) g(\alpha, e^{-\frac{v}{\kappa}}\beta) e^{i\alpha v}.$$

By analyticity of $g(\alpha, e^{-v}\beta)$ in v we have

$$g(\alpha, e^{-\frac{v}{\kappa}}\beta) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\kappa^n n!} v^n (\beta\partial_\beta)^n g(\alpha, \beta).$$

Inserting this into the previous equation and using that the series is uniformly convergent on the compact set K_f we get

$$(f *_\kappa g)(\alpha, \beta) = \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n}{\kappa^n n!} \int dv \, \tilde{f}(v, \beta) v^n (\beta \partial_\beta)^n g(\alpha, \beta) \, e^{i\alpha v} \, .$$

Now, use

$$\frac{1}{\sqrt{2\pi}} \int \tilde{f}(v, \beta) v^n e^{i\alpha v} = (-i\partial_\alpha)^n f(\alpha, \beta),$$

to conclude the proof. $\qquad\square$

5 Conclusions

The star product formulation of the κ-Minkowski algebra presented in this paper has potential advantages with regard to future developments. It is a basis-independent construction realized as a function space with a richer structure than the algebraic version, and with a simpler analytic form of the product than in previous approaches.

We consider it as first step towards the construction of a geometry on κ-Minkowski space in the sense of spectral triples. A primary goal will be to study the equivariant representations of the algebra $\mathcal{B}$ and to look for equivariant Dirac operators. The existence of the invariant twisted trace on $\mathcal{B}$ suggests that the geometry of κ-Minkowski space might be closer to the case of quantum groups (q-deformations) than originally believed. In particular, the failure of the spectral triple construction for the compactified version of κ-Minkowski space is possibly related to this fact, and the remedy might be to look for twisted spectral geometries.

Furthermore, there are interesting relations between the star product formulation of κ-Minkowski space and the deformations of Rieffel [14] determined by actions of $\mathbb{R}^d$. We postpone the discussion of these issues, as well as extensions to higher dimensions, to a future publication.

6 Appendix

The purpose of this appendix is to show that the definitions (52) and (61) of multiplication and inversion on $\mathcal{C}$ are independent of the choice of the functions functions χ_f and χ_F^1 satisfying the stated properties and to prove (55), (56) and (65).

The support of $m_* F$.

Let $F \in \mathcal{C} \otimes \mathcal{C}$. First, observe that by the definition (52) of m_* and the ensuing convergence arguments we have, for $\varphi \in \mathcal{S}(\mathbb{R}^2)$,

$$m_* F(\varphi) = \int d\alpha' \int dv' \int d\beta d\alpha \chi_F^1(v') F(\alpha + \alpha', \alpha, \beta, e^{-v'}\beta) \varphi(\alpha, \beta) e^{-i\alpha'v'} \, . \qquad (109)$$

For fixed $v', \beta \in \mathbb{R}$ and $\xi, \eta \in \mathcal{S}(\mathbb{R})$ we have

$$\int d\alpha' d\alpha F(\alpha + \alpha', \alpha, \beta, e^{-v'}\beta) \mathcal{F}\xi(\alpha') \mathcal{F}\eta(\alpha) = \int du du' \tilde{F}(u, u', \beta, e^{-v'}\beta) \xi(u) \eta(u + u') \, .$$

This vanishes if $\eta(u + u') = 0$ for all $(u, u') \in K_F$. Since this holds for arbitrary $\xi \in \mathcal{S}(\mathbb{R})$ it follows that

$$\int d\alpha F(\alpha + \alpha', \alpha, \beta, e^{-v'}\beta)\tilde{\varphi}(\alpha, \beta) = 0\,,$$

if $\varphi(u + u', \beta) = 0$ for all $(u, u') \in K_F$. Hence we get from (109) that $\widetilde{m_* F}(\varphi) = m_* F(\tilde{\varphi}) = 0$ if $\varphi(u + u', \beta)$ vanishes for $(u, u') \in K_F$ for arbitrary β. This proves (55).

Independence of the χ-functions.

Let $f \in \mathcal{C}$ and write

$$f^* = f_{1R}^* + f_{2R}^*\,,$$

where f_{1R}^* and f_{2R}^* are given by (62) and (63), respectively, with $\zeta(\alpha')$ replaced by $\zeta_R(\alpha') = \zeta_1(\frac{\alpha'}{R})$, and where ζ_1 is a smooth function of compact support that equals 1 on a neighborhood of 0. Choosing N in (63) sufficiently large, it follows from (48) that f_{2R}^* converges to 0 uniformly on compact subsets of $\mathbb{R}^2$ as $R \to \infty$. Hence, f_{1R}^* converges uniformly to f^* on compact subsets of $\mathbb{R}^2$. As the reader may easily verify, this also holds if we set $\zeta_1 = \mathcal{F}(\zeta)$, where ζ is a smooth function with support contained in $[-1, 1]$ such that $\int_{-\infty}^{\infty} \zeta(v)dv = 1$, since in this case

$$\zeta_R(v) = R\mathcal{F}(\zeta(Rv))$$

converges uniformly to 1 on compact subsets of $\mathbb{R}$ as $R \to \infty$. With this choice of ζ_R we have

$$f_{1R}^*(\alpha, \beta) = \frac{1}{\sqrt{2\pi}} \int dv \chi_f(-v) \int du\, R\zeta(R(v - u))\bar{\tilde{f}}(-u, e^{-v}\beta)\, e^{i\alpha u}\,.$$

Since the support of $u \to \zeta(Ru)$ is contained in $[-\frac{1}{R}, \frac{1}{R}]$ it follows that the last integral vanishes for all v outside any given distance $\delta > 0$ from $-K_f$ if $R > \frac{1}{\delta}$. This proves that the integral defining f^* only depends on the values of χ_f in any neighborhood of K_f as desired.

The proof that $m_* F$, $F \in \mathcal{C}_2$, only depends on the values of χ_F^1 in any neighborhood of the projection of K_F onto the first axis is essentially identical to the preceding argument and we skip further details.

Associativity of the product.

We consider m_* given by (52) and want to verify the relation (56). For $G \in \mathcal{C}_3$ we have by (52)

$$(m_* \otimes 1)G(\alpha_1, \alpha_2, \beta_1, \beta_2) = \frac{1}{2\pi} \int d\alpha_1' \int dv_1\, \chi_G^1(v_1)G(\alpha_1 + \alpha_1', \alpha_1, \alpha_2, \beta_1, e^{-v_1}\beta_1, \beta_2)e^{-i\alpha_1' v_1}$$

and

$$m_*(m_* \otimes 1)G(\alpha, \beta) = \frac{1}{(2\pi)^2} \int d\alpha_2' \int dv_2 \int d\alpha_1' \int dv_1$$

$$\chi_G^1(v_1)\chi_G^{++}(v_2)G(\alpha + \alpha_1' + \alpha_2', \alpha + \alpha_2', \alpha, \beta, e^{-v_1}\beta, e^{-v_2}\beta)e^{-i(\alpha_1' v_1 + \alpha_2' v_2)}\,, \tag{110}$$

where χ_G^{++} is a smooth function of compact support that equals 1 on a neighborhood of the set $\{v_1 + v_2 \mid (v_1, v_2, v_3) \in K_G \text{ for some } v_3 \in \mathbb{R}\}$. Similarly, we get

$$m_*(1 \otimes m_*)G(\alpha, \beta) = \frac{1}{(2\pi)^2} \int d\alpha_1' \int dv_1 \int d\alpha_2' \int dv_2$$

$$\chi_G^1(v_1)\chi_G^2(v_2)G(\alpha + \alpha_1', \alpha + \alpha_2', \alpha, \beta, e^{-v_1}\beta, e^{-(v_1 + v_2)}\beta)e^{-i(\alpha_1' v_1 + \alpha_2' v_2)} \tag{111}$$

Now, rewrite (110) as

$$
m_*(m_* \otimes 1)G(\alpha, \beta) = \frac{1}{(2\pi)^2} \int d\alpha_2' \int dv_2 \int d\alpha_1' \int dv_1
$$

$$
\chi_G^1(v_1)\chi_G^{++}(v_2)G(\alpha + \alpha_1', \alpha + \alpha_2', \alpha, \beta, e^{-v_1}\beta, e^{-v_2}\beta)e^{-i(\alpha_1' v_1 + \alpha_2'(v_2 - v_1))}
$$

and insert convergence factors $\zeta_R(\alpha_1')\zeta_R(\alpha_2')$ to justify interchange of integrations to obtain

$$
m_*(m_* \otimes 1)G(\alpha, \beta) = \frac{1}{(2\pi)^2} \int d\alpha_1' \int dv_1 \int d\alpha_2' \int dv_2
$$

$$
\chi_G^1(v_1)\chi_G^{++}(v_1 + v_2)G(\alpha + \alpha_1', \alpha + \alpha_2', \alpha, \beta, e^{-v_1}\beta, e^{-(v_1+v_2)}\beta)e^{-i(\alpha_1' v_1 + \alpha_2' v_2)} .
$$

By an argument similar to the one proving independence of f^* on the choice of χ_f above, we may in this integral replace the function $\chi_G^1(v_1)\chi_G^{++}(v_1 + v_2)$ by any smooth function of compact support that equals 1 on a neighborhood of the set $\{(v_1, v_2) \mid (v_1, v_2, v_3) \in K_G$ for some $v_3 \in \mathbb{R}\}$. Since this holds for the function $\chi_G^1(v_1)\chi_G^2(v_2)$ we conclude that the integrals (110) and (111) are equal as desired.

The *-operation is an antihomomorphisn.

Let $F \in \mathcal{C}_2$ and let $\chi_F^+, \chi_F^1, \chi_F^2$ denote smooth functions of compact support that equal 1 on the α-support of m_*F and on the projections of K_F onto the first and second coordinate axis, respectively. Using definitions (52) and (61) we then have

$$(m_*F)^*(\alpha, \beta) = \frac{1}{(2\pi)^2} \int d\alpha_2' \int dv_2 \int d\alpha_1' \int dv_1$$
$$\chi_F^1(v_1)\chi_F^+(-v_2)\bar{F}(\alpha + \alpha_1', \alpha + \alpha_2', e^{-v_2}\beta, e^{-(v_1+v_2)}\beta)e^{i\alpha_1'v_1 - i\alpha_2'(v_1+v_2)} \tag{112}$$

For the right-hand side of (65), on the other hand, we get

$$m_*((F^*)^\wedge)(\alpha, \beta) =$$
$$\frac{1}{(2\pi)^3} \int d\alpha' \int dv \int d\alpha_2' \int dv_2 \int d\alpha_1' \int dv_1 \, \chi_F^2(-v)\chi_F^2(-v_2)\chi_F^1(-v_1) \tag{113}$$
$$\bar{F}(\alpha + \alpha_1', \alpha_2', e^{-(v_1+v)}\beta, e^{-v_2}\beta)e^{-i(\alpha_1'v_1 + \alpha_2'v_2) + i\alpha v_2}e^{i\alpha'(v_2 - v)} \,.$$

Inserting convergence factors $\zeta_{R_1}(\alpha_1')\zeta_{R_2}(\alpha_2')\zeta_R(\alpha')$ into the last integral we recover its value in the limit $R, R_1, R_2 \to \infty$ by the same arguments as above. By performing the α'-integration first in the regularized integral we obtain

$$\frac{1}{(2\pi)^{5/2}} \int dv \int d\alpha_2' \int dv_2 \int d\alpha_1' \int dv_1\chi_F^2(-v)\chi_F^2(-v_2)\chi_F^1(-v_1)$$
$$\zeta_{R_2}(\alpha_2')\zeta_{R_1}(\alpha_1')\bar{F}(\alpha + \alpha_1', \alpha_2', e^{-(v_1+v)}\beta, e^{-v}\beta)e^{-i(\alpha_1'v_1 + \alpha_2'v_2) + i\alpha v}\mathcal{F}(\zeta_R)(v - v_2) \,,$$

and in the limit $R \to \infty$ this gives

$$\frac{1}{(2\pi)^2} \int d\alpha_2' \int dv_2 \int d\alpha_1' \int dv_1 \, \chi_F^2(-v_2)^2\chi_F^1(-v_1)$$
$$\zeta_{R_2}(\alpha_2')\zeta_{R_1}(\alpha_1')\bar{F}(\alpha + \alpha_1', \alpha_2', e^{-(v_1+v_2)}\beta, e^{-v_2}\beta)e^{-i(\alpha_1'v_1 + \alpha_2'v_2) + i\alpha v_2} \,.$$

A simple change of variables now yields

$$m_*((F^*)^\wedge)(\alpha, \beta) = \lim_{R_1, R_2 \to \infty} \frac{1}{(2\pi)^2} \int d\alpha_2' \int dv_2 \int d\alpha_1' \int dv_1 \, \chi_F^2(-v_1 - v_2)^2\chi_F^1(v_1)$$
$$\zeta_{R_2}(\alpha_2')\zeta_{R_1}(\alpha_1')\bar{F}(\alpha + \alpha_1', \alpha + \alpha_2', e^{-v_2}\beta, e^{-(v_1+v_2)}\beta)e^{i\alpha_1'v_1 - i\alpha_2'(v_1+v_2)} \,,$$

Repeating previous arguments we see by choosing ζ_R such that $\mathcal{F}(\zeta_R)$ has support in $[\frac{1}{R}, \frac{1}{R}]$ that in the limit above the function $\chi_F^2(-v_1 - v_2)^2\chi_F^1(v_1)$ can be replaced by any smooth function of compact support that equals 1 on a neighborhood of the set $\{(v_1, v_2) \mid (v_1, -v_1 - v_2) \in K_F\}$ without changing the value of the limit. Since this holds, in particular, for the function $\chi_F^1(v_1)\chi_F^+(-v_2)$ we conclude that the limit equals (112). This proves (65).

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